ARBITRARY RECURSION

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One Recursive Call

arbitrary, possibly non-terminating, recursive "definition" (e.g. a program-mode function):

$$f(\bar{x}) \stackrel{?}{=} \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ c(\bar{x}, f(\bar{d}(\bar{x}))) \qquad \bar{x} = (x_1, \ldots, x_n) \qquad \bar{d}(\bar{x}) = (d_1(\bar{x}), \ldots, d_n(\bar{x})) \qquad n > 0$$

$$a \in \mathcal{U}^n \qquad b : \mathcal{U}^n \to \mathcal{U}^m \qquad d_i : \mathcal{U}^n \to \mathcal{U} \qquad c : \mathcal{U}^n \times \mathcal{U}^m \to \mathcal{U}^m \qquad m > 0 \ \text{(number of results)}$$

$$t(\bar{x}) \triangleq [\exists k \in \mathbb{N}.\ a(\bar{d}^k(\bar{x}))] \quad \text{— } f \text{ terminates on } \bar{x}$$

always-terminating "version" of f (a logic-mode function, if a, b, c, d are logic-mode):

$$\hat{f}(\bar{x}) \triangleq \underline{\text{if}}\ t(\bar{x})\ \underline{\text{then}}\ [\underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ c(\bar{x}, \hat{f}(\bar{d}(\bar{x})))]\ \underline{\text{else}}\ \ldots$$ any value (irrelevant)

$$\mu_{\hat{f}}(\bar{x}) \triangleq \underline{\text{if}}\ t(\bar{x})\ \underline{\text{then}}\ \min\{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\}\ \underline{\text{else}}\ \ldots$$ any natural number (irrelevant) $\qquad \mu_{\hat{f}} : \mathcal{U}^n \to \mathbb{N}$

$$\prec_{\hat{f}} \triangleq < \subseteq \mathbb{N} \times \mathbb{N}$$

$\vdash$ $\boxed{\mu\text{-end}}$ $t(\bar{x}) \Longrightarrow a(\bar{d}^{\mu_{\hat{f}}(\bar{x})}(\bar{x}))$

$$t(\bar{x}) \xrightarrow{\delta_{\mu_{\hat{f}}}} \mu_{\hat{f}}(\bar{x}) = \min\{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow \mu_{\hat{f}}(\bar{x}) \in \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow a(\bar{d}^{\mu_{\hat{f}}(\bar{x})}(\bar{x}))$$

QED

$\vdash$ $\boxed{\mu\text{-min}}$ $a(\bar{d}^l(\bar{x})) \Longrightarrow l \geq \mu_{\hat{f}}(\bar{x})$

$$a(\bar{d}^l(\bar{x})) \xrightarrow{\delta_t} t(\bar{x}) \xrightarrow{\delta_{\mu_{\hat{f}}}} \mu_{\hat{f}}(\bar{x}) = \min\{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow l \geq \mu_{\hat{f}}(\bar{x})$$

$$a(\bar{d}^l(\bar{x})) \longrightarrow l \in \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow l \geq \mu_{\hat{f}}(\bar{x})$$

QED

$\vdash$ $\boxed{\tau_{\hat{f}}}$ $t(\bar{x}) \wedge \neg a(\bar{x}) \Longrightarrow \mu_{\hat{f}}(\bar{d}(\bar{x})) \prec_{\hat{f}} \mu_{\hat{f}}(\bar{x})$

$$t(\bar{x}) \xrightarrow{\mu\text{-end}} a(\bar{d}^{\mu_{\hat{f}}(\bar{x})}(\bar{x})) \longrightarrow a(\bar{d}^{\mu_{\hat{f}}(\bar{x})-1}(\bar{d}(\bar{x}))) \xrightarrow[\mu\text{-min}]{l := \mu_{\hat{f}}(\bar{x})-1,\ \bar{x} := \bar{d}(\bar{x})} \mu_{\hat{f}}(\bar{x}) - 1 \geq \mu_{\hat{f}}(\bar{d}(\bar{x})) \longrightarrow \mu_{\hat{f}}(\bar{x}) \geq \mu_{\hat{f}}(\bar{d}(\bar{x})) + 1 > \mu_{\hat{f}}(\bar{d}(\bar{x})) \xrightarrow{\delta_{\prec_{\hat{f}}}} \mu_{\hat{f}}(\bar{d}(x)) \prec_{\hat{f}} \mu_{\hat{f}}(\bar{x})$$

$$\neg a(\bar{x}) \longrightarrow \neg a(\bar{d}^0(\bar{x})) \longrightarrow \mu_{\hat{f}}(\bar{x}) \neq 0$$

QED

alternative recursive definition of the measure function:

$\varphi(u) \triangleq \underline{\text{if}}\ u \in N\ \underline{\text{then}}\ u\ \underline{\text{else}}\ 0$ — fixing function for N

$\varepsilon_t(\bar{x}) \triangleq \varepsilon k.\ a(\bar{d}^{\varphi(k)}(\bar{x}))$ — witness of t, for slightly modified definition $t(\bar{x}) \triangleq [\exists k.\ a(\bar{d}^{\varphi(k)}(\bar{x}))]$

$\nu(\bar{x},k) \triangleq \underline{\text{let}}\ \tilde{k} = \varphi(k)\ \underline{\text{in}}\ \underline{\text{if}}\ a(\bar{d}^{\tilde{k}}(\bar{x})) \vee \tilde{k} \geq \varphi(\varepsilon_t(\bar{x}))\ \underline{\text{then}}\ \tilde{k}\ \underline{\text{else}}\ \nu(\bar{x}, \tilde{k}+1)$

$\mu_\nu(\bar{x},k) \triangleq \varphi(\varepsilon_t(\bar{x}) - \varphi(k))$ $\prec_\nu \triangleq < \subseteq N \times N$

} recursively find min k such that $a(\bar{d}^k(\bar{x}))$, if $t(\bar{x})$; stop at $\varphi(\varepsilon_t(\bar{x}))$ anyhow; min k is always found if $t(\bar{x})$, because min $k \leq \varphi(\varepsilon_t(\bar{x}))$

$\vdash \boxed{\tau_\nu}$ $\neg a(\bar{d}^{\varphi(k)}(\bar{x})) \wedge \varphi(k) < \varphi(\varepsilon_t(\bar{x})) \Rightarrow \mu_\nu(\bar{x}, \varphi(k)+1) \prec_\nu \mu_\nu(\bar{x},k)$

$\mu_\nu(\bar{x},\varphi(k)+1) \overset{\delta\mu_\nu}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(\varphi(k)+1)) \underset{\varphi(k)\in N}{\overset{\delta\varphi}{=}} \varphi(\varepsilon_t(\bar{x}) - \varphi(k) - 1) \overset{\delta\varphi}{=} \varepsilon_t(\bar{x}) - \varphi(k) - 1$

$\varphi(k) < \varphi(\varepsilon_t(\bar{x})) \longrightarrow \varphi(\varepsilon_t(\bar{x})) \neq 0 \xrightarrow{\delta\varphi} \varepsilon_t(\bar{x}) \in N \longrightarrow \varphi(k) < \varepsilon_t(\bar{x}) \longrightarrow \varepsilon_t(\bar{x}) - \varphi(k) > 0$

$\mu_\nu(\bar{x},k) \overset{\delta\mu_\nu}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(k)) \underset{\delta\varphi}{=} \varepsilon_t(\bar{x}) - \varphi(k) \longrightarrow \mu_\nu(\bar{x},\varphi(k)+1) \prec_\nu \mu_\nu(\bar{x},k)$ $\delta\prec_\nu$

QED

$\vdash \boxed{\nu N}$ $\nu(\bar{x},k) \in N$

induct ν

base) $\delta_\nu \longrightarrow \nu(\bar{x},k) = \varphi(k)$; $\delta\varphi \longrightarrow \varphi(k) \in N$ $\longrightarrow \nu(\bar{x},k) \in N$

step) $\delta_\nu \longrightarrow \nu(\bar{x},k) = \nu(\bar{x}, \varphi(k)+1)$; IH $\longrightarrow \nu(\bar{x},\varphi(k)+1) \in N$ $\longrightarrow \nu(\bar{x},k) \in N$

QED

$\mu_{\hat{f}}(\bar{x}) \triangleq \nu(\bar{x},0)$ — alternative definition of $\mu_{\hat{f}}$ $\mu_{\hat{f}} : U^n \to N$ $(\Leftarrow \nu N)$

⊢ $\boxed{\nu\text{-end}}$ $t(\bar{x}) \wedge k \in \mathbb{N} \wedge k \le \varphi(\varepsilon_t(\bar{x})) \implies a(\bar{d}^{\nu(\bar{x},k)}(\bar{x}))$

induct ν

base) $a(\bar{d}^{\varphi(k)}(\bar{x})) \vee \varphi(k) \geq \varphi(\varepsilon_t(\bar{x}))$

$a(\bar{d}^{\varphi(k)}(\bar{x})) \xrightarrow{\delta_\nu} \nu(\bar{x},k) = \varphi(k) \longrightarrow a(\bar{d}^{\nu(\bar{x},k)}(\bar{x}))$

$\varphi(k) \geq \varphi(\varepsilon_t(\bar{x})) \xrightarrow{\delta_\varphi} k \geq \varphi(\varepsilon_t(\bar{x}))$

$k \in \mathbb{N}$

δ_φ

$k \le \varphi(\varepsilon_t(\bar{x})) \longrightarrow k = \varphi(\varepsilon_t(\bar{x}))$

δ_ν $\nu(\bar{x},k) = \varphi(k) = k$

$t(\bar{x}) \xrightarrow{\delta_t} a(\bar{d}^{\varphi(\varepsilon_t(\bar{x}))}(\bar{x})) \longrightarrow a(\bar{d}^{k}(\bar{x})) \longrightarrow a(\bar{d}^{\nu(\bar{x},k)}(\bar{x}))$

step) $\neg a(\bar{d}^{\varphi(k)}(\bar{x})) \wedge \varphi(k) < \varphi(\varepsilon h . a(\bar{d}^{h}(\bar{x}))) \xrightarrow{\delta_\nu} \nu(\bar{x},k) = \nu(\bar{x},\varphi(k)+1)$

$\varphi(k)+1 \le \varphi(\varepsilon h . a(\bar{d}^{h}(\bar{x})))$

δ_φ

$\varphi(k)+1 \in \mathbb{N}$

$t(\bar{x}) \xrightarrow{IH} a(\bar{d}^{\nu(\bar{x},\varphi(k)+1)}(\bar{x})) \longrightarrow a(\bar{d}^{\nu(\bar{x},k)}(\bar{x}))$

QED

⊢ $\boxed{\mu\text{-end}}$ $t(\bar{x}) \implies a(\bar{d}^{\mu_{\hat{f}}(\bar{x})}(\bar{x}))$ — alternative proof for the alternative definition of $\mu_{\hat{f}}$

$t(\bar{x})$

$0 \in \mathbb{N}$

$\delta_\varphi \longrightarrow 0 \le \varphi(\varepsilon_t(\bar{x}))$

$\xrightarrow{\nu\text{-end},\ k := 0} a(\bar{d}^{\nu(\bar{x},0)}(\bar{x})) \xrightarrow{\delta_{\mu_{\hat{f}}}} a(\bar{d}^{\mu_{\hat{f}}(\bar{x})}(\bar{x}))$

QED

⊢ $\boxed{\nu\text{-min}}$ $l \in \mathbb{N} \wedge k \in \mathbb{N} \wedge l \geqslant k \wedge a(d^{l}(\bar{x})) \Longrightarrow l \geqslant \nu(\bar{x}, k)$

induct ν

base) $a(\bar{d}^{\varphi(k)}(\bar{x})) \vee \varphi(k) \geqslant \varphi(\varepsilon_t(\bar{x})) \xrightarrow{\delta\nu} \nu(\bar{x}, k) = \varphi(k) = k$

$k \in \mathbb{N} \xrightarrow{\delta_\varphi}$

$l \geqslant k \longrightarrow l \geqslant \nu(\bar{x}, k)$

step) $\neg a(\bar{d}^{\varphi(k)}(\bar{x})) \wedge \varphi(k) < \varphi(\varepsilon_t(\bar{x})) \xrightarrow{\delta_\nu} \nu(\bar{x}, k) = \nu(\bar{x}, \varphi(k)+1)$

$a(d^{l}(\bar{x})) \longrightarrow l \neq \varphi(k)$

$k \in \mathbb{N} \xrightarrow{\delta_\varphi} l \neq k$

$l \in \mathbb{N} \longrightarrow l \geqslant k+1 \stackrel{\delta_\varphi}{=} \varphi(k)+1$

$l \geqslant k$

$\delta_\varphi \rightarrow \varphi(k)+1 \in \mathbb{N}$

$\xrightarrow{IH} l \geqslant \nu(\bar{x}, \varphi(k)+1) \longrightarrow l \geqslant \nu(\bar{x}, k)$

QED

⊢ $\boxed{\mu\text{-min}}$ $a(\bar{d}^{l}(\bar{x})) \Longrightarrow l \geqslant \mu_{\hat{f}}(\bar{x})$ — alternative proof for the alternative definition of $\mu_{\hat{f}}$

$l \in \mathbb{N}$

$0 \in \mathbb{N}$

$l \geqslant 0$

$a(\bar{d}^{l}(\bar{x}))$

$\xrightarrow{\nu\text{-min} \quad k:=0} l \geqslant \nu(\bar{x}, 0) \xrightarrow{\delta_{\mu\hat{f}}} l \geqslant \mu_{\hat{f}}(\bar{x})$

QED

$\tau_{\hat{f}}$ proved as before using μ-end and μ-min