DOMAIN MAPPINGS

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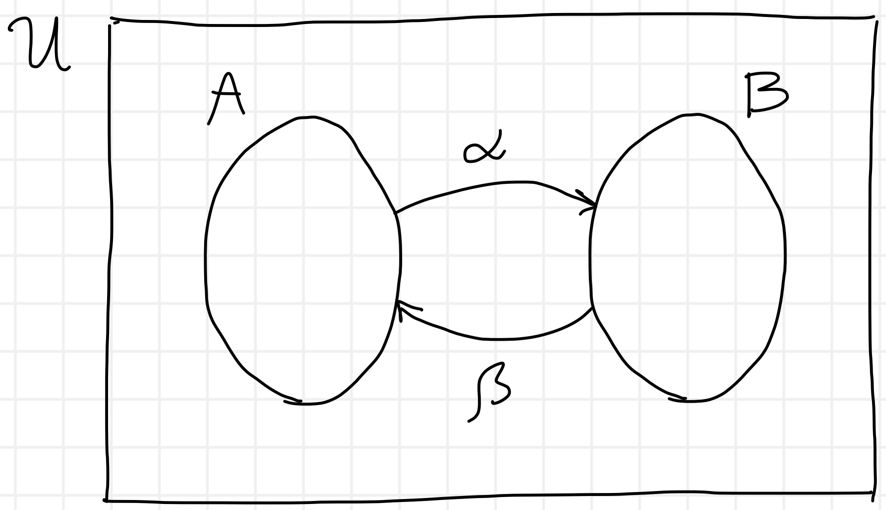
Domain Mapping

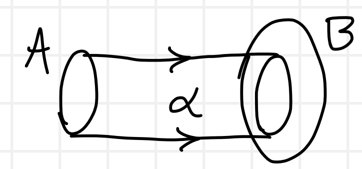


$$\left.\begin{array}{l} A \subseteq \mathcal{U} \\ B \subseteq \mathcal{U} \end{array}\right\} \text{domains (unary predicates)}$$

$$\left.\begin{array}{l} \alpha : \mathcal{U} \to \mathcal{U} \\ \beta : \mathcal{U} \to \mathcal{U} \end{array}\right\} \text{conversions (unary functions)}$$

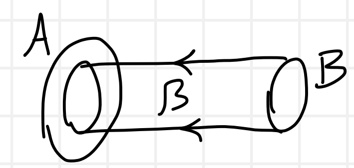
required conditions

- $\boxed{\alpha A}$ $\forall a \in A.\ \alpha(a) \in B$ — α maps A to B



$\alpha(A) \subseteq B$

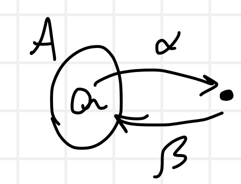
- $\boxed{\beta B}$ $\forall b \in B.\ \beta(b) \in A$ — β maps B to A



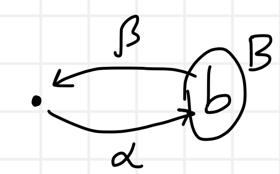
$\beta(B) \subseteq A$

optional conditions

- $\boxed{\beta\alpha}$ $\forall a \in A.\ \beta(\alpha(a)) = a$ — β is left inverse of α over A; α is right inverse of β over A



- $\boxed{\alpha\beta}$ $\forall b \in B.\ \alpha(\beta(b)) = b$ — α is left inverse of β over B; β is right inverse of α over B

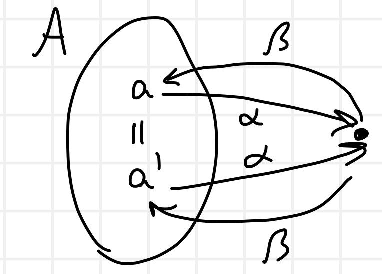


$\beta\alpha \implies \vdash \boxed{\alpha i}\ \forall a, a' \in A.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — α injective on A

$a \in A \qquad \alpha(a) = \alpha(a') \qquad a' \in A$

$\beta\alpha \qquad \underset{\underset{a}{\shortparallel}}{\beta(\alpha(a))} = \underset{\underset{a'}{\shortparallel}}{\beta(\alpha(a'))} \qquad \beta\alpha$

QED

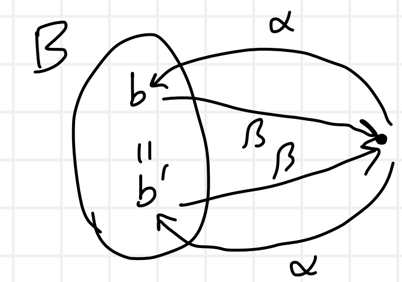


$\alpha\beta \implies \vdash \boxed{\beta i}\ \forall b, b' \in B.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — β injective on B

$b \in B \qquad \beta(b) = \beta(b') \qquad b' \in B$

$\alpha\beta \qquad \underset{\underset{b}{\shortparallel}}{\alpha(\beta(b))} = \underset{\underset{b'}{\shortparallel}}{\alpha(\beta(b'))} \qquad \alpha\beta$

QED



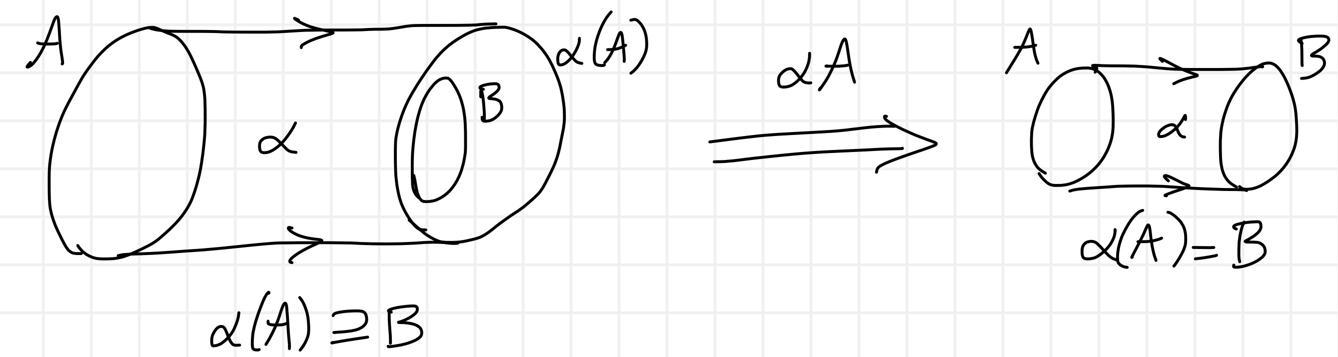
$\alpha A \wedge \alpha\beta \implies \vdash \boxed{\alpha s}\ \forall b \in B.\ \exists a \in A.\ b = \alpha(a)$ — α surjective on B from A

$b \in B \xrightarrow{\alpha\beta} \alpha(\beta(b)) = b$

$\beta B \qquad a \triangleq \beta(b) \nearrow \overset{\shortparallel}{\alpha(a)}$

$a \in A$

QED



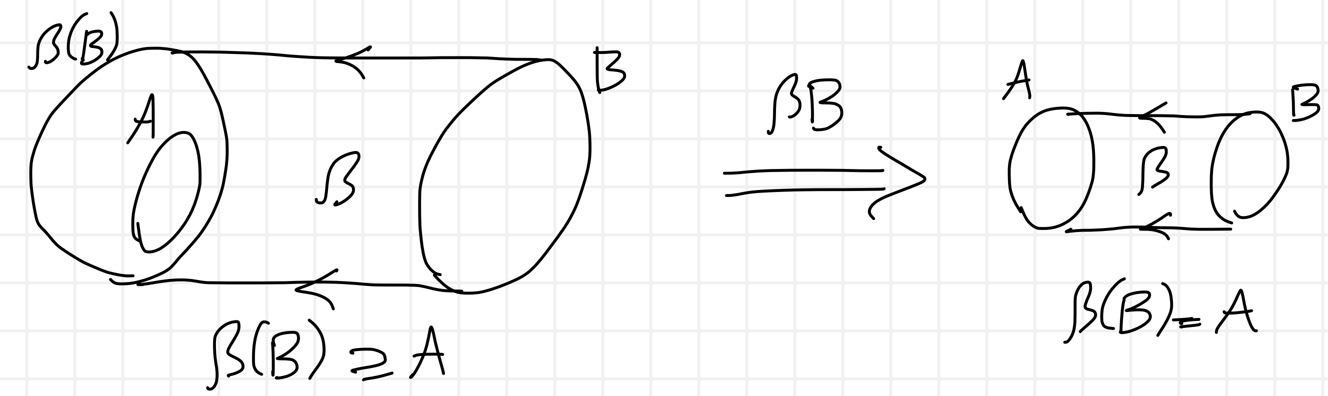
$\beta B \wedge \beta\alpha \implies \vdash \boxed{\beta s}\ \forall a \in A.\ \exists b \in B.\ a = \beta(b)$ — β surjective on A from B

$a \in A \xrightarrow{\beta\alpha} \beta(\alpha(a)) = a$

$\alpha A \qquad b \triangleq \alpha(a) \nearrow \overset{\shortparallel}{\beta(b)}$

$b \in B$

QED



$A \underset{\beta}{\overset{\alpha}{\rightarrow}} B \triangleq \alpha A \wedge \beta B$ — α and β are mappings between A and B

$A \underset{\beta}{\overset{\alpha}{\rightarrowtail}} B \triangleq \alpha A \wedge \beta B \wedge \beta\alpha$ — α is an injection from A to B with left inverse β

$A \underset{\beta}{\overset{\alpha}{\twoheadrightarrow}} B \triangleq \alpha A \wedge \beta B \wedge \alpha\beta$ — α is a surjection from A to B with right inverse β

$A \underset{\beta}{\overset{\alpha}{\leftrightarrow}} B \triangleq \alpha A \wedge \beta B \wedge \beta\alpha \wedge \alpha\beta$ — α and β are mutually inverse isomorphisms between A and B

$\vdash A \underset{\beta}{\overset{\alpha}{\rightarrowtail}} B \Longleftrightarrow A \underset{\beta}{\overset{\alpha}{\rightarrow}} B \wedge \beta\alpha$

$\vdash A \underset{\beta}{\overset{\alpha}{\twoheadrightarrow}} B \Longleftrightarrow A \underset{\beta}{\overset{\alpha}{\rightarrow}} B \wedge \alpha\beta$

$\vdash A \underset{\beta}{\overset{\alpha}{\leftrightarrow}} B \Longleftrightarrow A \underset{\beta}{\overset{\alpha}{\rightarrow}} B \wedge \beta\alpha \wedge \alpha\beta \Longleftrightarrow A \underset{\beta}{\overset{\alpha}{\rightarrowtail}} B \wedge A \underset{\beta}{\overset{\alpha}{\twoheadrightarrow}} B$

$\vdash A \underset{\beta}{\overset{\alpha}{\rightarrow}} B \Longleftrightarrow B \underset{\alpha}{\overset{\beta}{\rightarrow}} A$

$\vdash A \underset{\beta}{\overset{\alpha}{\rightarrowtail}} B \Longleftrightarrow B \underset{\alpha}{\overset{\beta}{\twoheadrightarrow}} A$

$\vdash A \underset{\beta}{\overset{\alpha}{\twoheadrightarrow}} B \Longleftrightarrow B \underset{\alpha}{\overset{\beta}{\rightarrowtail}} A$

$\vdash A \underset{\beta}{\overset{\alpha}{\leftrightarrow}} B \Longleftrightarrow B \underset{\alpha}{\overset{\beta}{\leftrightarrow}} A$

Guards

conditions
$$\left\{\begin{array}{lll}\boxed{GA} & \gamma_A = \mathcal{U} & \text{— } A \text{ well-defined everywhere} \\ \boxed{GB} & \gamma_B = \mathcal{U} & \text{— } B \text{ well-defined everywhere} \\ \boxed{G\alpha} & \gamma_\alpha \supseteq A & \text{— } \alpha \text{ well-defined at least over } A \\ \boxed{G\beta} & \gamma_\beta \supseteq B & \text{— } \beta \text{ well-defined at least over } B \end{array}\right.$$

$\vdash \boxed{\surd \alpha A} \quad \omega_{\alpha A}(a)$

$$\omega_{\alpha A}(a) = \Big[\underbrace{\cancel{\gamma_A(a)}}_{GA} \wedge \Big[a \in A \underset{G\alpha}{\Longrightarrow} \cancel{\gamma_\alpha(a)} \wedge \underbrace{\cancel{\gamma_B(\alpha(a))}}_{GB}\Big]\Big]$$

QED

$\vdash \boxed{\surd \beta B} \quad \omega_{\beta B}(b)$

$$\omega_{\beta B}(b) = \Big[\underbrace{\cancel{\gamma_B(b)}}_{GB} \wedge \Big[b \in B \underset{G\beta}{\Longrightarrow} \cancel{\gamma_\beta(b)} \wedge \underbrace{\cancel{\gamma_A(\beta(b))}}_{GA}\Big]\Big]$$

QED

$\vdash \boxed{\surd \beta \alpha} \quad \omega_{\beta\alpha}(a)$

$$\omega_{\beta\alpha}(a) = \Big[\underbrace{\cancel{\gamma_A(a)}}_{GA} \wedge \Big[a \in A \overset{G\alpha}{\Longrightarrow} \cancel{\gamma_\alpha(a)} \wedge \cancel{\gamma_\beta(\alpha(a))}\Big]\Big]$$

$$a \in A \xrightarrow{\alpha A} \alpha(a) \in B \xrightarrow{G\beta} \gamma_\beta(\alpha(a))$$

QED

$\vdash \boxed{\surd \alpha \beta} \quad \omega_{\alpha\beta}(b)$

$$\omega_{\alpha\beta}(b) = \Big[\underbrace{\cancel{\gamma_B(b)}}_{GB} \wedge \Big[b \in B \overset{G\beta}{\Longrightarrow} \cancel{\gamma_\beta(b)} \wedge \cancel{\gamma_\alpha(\beta(b))}\Big]\Big]$$

$$b \in B \xrightarrow{\beta B} \beta(b) \in A \xrightarrow{G\alpha} \gamma_\alpha(\beta(b))$$

QED

$\dashrightarrow \in \{\rightarrow, \rightarrowtail, \twoheadrightarrow, \leftrightarrow\}$

$G[A \overset{\alpha}{\underset{\beta}{\dashrightarrow}} B] \triangleq GA \wedge GB \wedge G\alpha \wedge G\beta$ — the constituents of $A \overset{\alpha}{\underset{\beta}{\dashrightarrow}} B$ satisfy the guard conditions

Generalization to Tuples

$A \subseteq \mathcal{U}^n \quad B \subseteq \mathcal{U}^m \quad \alpha : \mathcal{U}^n \to \mathcal{U}^m \quad \beta : \mathcal{U}^m \to \mathcal{U}^n$

everything works the same as in the unary case

Variant: Unconditional Theorems

$\boxed{\beta\alpha'}$ $\forall a.\ \beta(\alpha(a)) = a$ — holds for $a \notin A$ too

$\boxed{\alpha\beta'}$ $\forall b.\ \alpha(\beta(b)) = b$ — holds for $b \notin B$ too

$\vdash \boxed{\alpha i'}$ $\forall a, a'.\ \alpha(a) = \alpha(a') \Rightarrow a = a'$ — holds for $a \notin A$ or $a' \notin A$ too

$$\alpha(a) = \alpha(a')$$
$$\downarrow$$
$$a \overset{\beta\alpha'}{=} \beta(\alpha(a)) = \beta(\alpha(a')) \overset{\beta\alpha'}{=} a'$$

QED

$\vdash \boxed{\beta i'}$ $\forall b, b'.\ \beta(b) = \beta(b') \Rightarrow b = b'$ — holds for $b \notin B$ or $b' \notin B$ too

$$\beta(b) = \beta(b')$$
$$\downarrow$$
$$b \overset{\alpha\beta'}{=} \alpha(\beta(b)) = \alpha(\beta(b')) \overset{\alpha\beta'}{=} b'$$

QED

$\vdash \boxed{\alpha s'}$ $\forall b.\ \exists a.\ b = \alpha(a)$ — holds for $b \notin B$ too but it may be $a \notin A$

$\alpha\beta' \rightarrow \alpha(\beta(b)) = b$

$a \triangleq \beta(b) \longrightarrow \alpha(a)$

QED

$\vdash \boxed{\beta s'}$ $\forall a.\ \exists b.\ a = \beta(b)$ — holds for $a \notin A$ too but it may be $b \notin B$

$\beta\alpha' \rightarrow \beta(\alpha(a)) = a$

$b \triangleq \alpha(a) \longrightarrow \beta(b)$

QED

making also αA and βB unconditional seems unnecessery: just have $A = B = \mathcal{U}$ instead