Fourier Coefficient Formalization in ACL2(r)

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Outline

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Orthogonality Relations of Trigonometric Functions

Theorem 1 (Orthogonality relations of trigonometric functions)

$$\int_{-L}^{L} \sin(m\frac{\pi}{L}x)\sin(n\frac{\pi}{L}x)dx = \begin{cases} 0, & \textit{if } m \neq n \vee m = n = 0 \\ L, & \textit{if } m = n \neq 0 \end{cases}$$

$$\int_{-L}^{L} \cos(m\frac{\pi}{L}x)\cos(n\frac{\pi}{L}x)dx = \begin{cases} 0, & \textit{if } m \neq n \\ L, & \textit{if } m = n \neq 0 \\ 2L, & \textit{if } m = n = 0 \end{cases}$$

$$\int_{-L}^{L} \sin(m\frac{\pi}{L}x)\cos(n\frac{\pi}{L}x)dx = 0$$

- Syntax is like `defun`:

```
(defun-std f (x1 ... xn)
  <body>)  ;; note that <body> does not need
           ;; to be classical!
```

- Proof obligation for the above `defun-std` form:

```
(implies (and (standardp x1) ... (standardp xn))
         (standardp <body>))
```

- Axiom added for the above `defun-std` form:

```
(implies (and (standardp x1) ... (standardp xn))
         (equal (f x1 ... xn)
                <body>))
```

Defthm-std

- The transfer principle is implemented in ACL2(r) with `defthm-std`.
  `(defthm-std name <body>)   ;; optionally, :hints etc.`

- Apply if the <body> is classical. Before attempting the proof, ACL2(r) adds a hypothesis of `(standardp x)` for all variables `x` in the <body>:

```
(implies (and (standardp x1) ... (standardp xk))
         <body>)
```

- Also apply to prove that a classical function returns standard values with standard inputs. Formally, if `f` is classical, then

```
(defthm-std name
  (implies (and (standardp x1) ... (standardp xk))
           (standardp (f x1 ... xk))))
```

FTC-2 Proof Procedure

FTC-2: If f' is a real-valued continuous function on $[a, b]$ and f is an antiderivative of f' on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

- Prove that f' returns real values on $[a, b]$.
- Prove that f' is continuous on $[a, b]$.
- Specify the real-valued antiderivative f of f' and prove that f' is the derivative of f on $[a, b]$.
- Formalize the integral of f' on $[a, b]$.
- Evaluate the integral of f' on $[a, b]$ in terms of f by applying the FTC-2.

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Riemann Integral

The Riemann integral of a function f' on an interval $[a, b]$ is the limit (if exists) of the Riemann sum of f' when partitioning $[a, b]$ into extremely small subintervals.

In non-standard analysis, the Riemann integral can be defined as the standard part of the Riemann sum (if limited) when partitioning $[a, b]$ into infinitesimal subintervals.

```
(defund-std strict-int-f-prime (a b)
  (if (and (inside-interval-p a (f-prime-domain))
           (inside-interval-p b (f-prime-domain))
           (< a b))
      (standard-part
       (riemann-f-prime (make-small-partition a b)))
    0))
```

Proof obligation: the Riemann sum is limited on $[a, b]$.

Riemann Integral

```
(defthm limited-riemann-f-prime-small-partition
  (implies (and (standardp a)
                (standardp b)
                (inside-interval-p a (f-prime-domain))
                (inside-interval-p b (f-prime-domain))
                (< a b))
           (i-limited
            (riemann-f-prime (make-small-partition a b)))))
```

The limited property of Riemann sums was proved in ACL2 community books for generic real-valued continuous unary functions [M. Kaufmann, 2000].

Riemann Integral

```
(defthm limited-riemann-f-prime-small-partition
  (implies (and (standardp a)
                (standardp b)
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                (inside-interval-p b (f-prime-domain))
                (< a b))
           (i-limited
            (riemann-f-prime (make-small-partition a b)))))
```

The limited property of Riemann sums was proved in ACL2 community books for generic real-valued continuous unary functions [M. Kaufmann, 2000].

Unfortunately, we are not allowed to functionally instantiate the lemma above for functions containing more than one variable (i.e., functions containing free variables) since the theorem we try to instantiate is non-classical and the functions we try to instantiate are classical [R. Gamboa & J. Cowles, 2007].

Functional Instantiation Issue

Example: Given an arbitrary classical function $f(x)$, it follows that

$$standardp(x) \Rightarrow standardp(f(x))$$

If we are allowed to substitute $\lambda(x).(x+y)$ into the formula above, we would conclude that

$$standardp(x) \Rightarrow standardp(x+y)$$

But this is false since the free variable y can be non-standard.

Limited Property of Riemann Sums

Theorem 2 (Limited property of Riemann sums)

If there exists finite values m and M such that

$$m \leq f(t) \leq M, \ \textit{for all } t \in [a, b]$$

Then the Riemann sum of f over $[a, b]$ *with any partition P is bounded by*

$$m(b-a) \leq \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq M(b-a)$$

where $t_i \in [x_{i-1}, x_i] \wedge x_0 = a \wedge x_n = b.$

Limited Property of Riemann Sums

Proof. From the hypothesis $m \leq f(t) \leq M$ for all $t \in [a, b]$, it follows that

$$\sum_{i=1}^{n} m(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M(x_i - x_{i-1})$$

where $t_i \in [x_{i-1}, x_i] \wedge x_0 = a \wedge x_n = b$.

$$\Rightarrow m\sum_{i=1}^{n}(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq M\sum_{i=1}^{n}(x_i - x_{i-1})$$

$$\Rightarrow m(b - a) \leq \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq M(b - a)$$

□

Problem

Given a specific real-valued continuous function f, it is usually easy to specify the bounds of f on a closed and bounded interval.

The problem becomes more challenging when applying to generic real-valued continuous functions since it is impossible to find either their minimum or maximum. However, the boundedness of these functions on a closed and bounded interval still holds by the extreme value theorem.

Extreme Value Theorem

Theorem 3 (Extreme value theorem)

Given any real-valued continuous function f defined on a closed and bounded interval $[a, b]$, there always exist numbers c and d in $[a, b]$ s.t.

$$f(c) \leq f(x) \leq f(d), \forall x \in [a, b]$$

The extreme value theorem was proved in ACL2 community books for unary functions only [J. Cowles & R. Gamboa, 2014]. We need to extend this theorem for functions with free variables.

Functional Instantiation with Free Variables

Add only one extra variable representing free variables to the constrained function and ignore this extra variable in the function definition.

```
(encapsulate
 ((rcfn-2 (x arg) t)
  (rcfn-2-domain () t))

 (local (defun rcfn-2 (x arg)
          (declare (ignore arg))
          (realfix x)))
 (local (defun rcfn-2-domain () (interval nil nil)))

 ... ;; Non-local theorems about rcfn-2 and rcfn-2-domain
 )
```

⇒ The proofs for the constrained function with main variables only are still applied for the new constrained function with the extra variable added.

Functional Instantiation with Free Variables

Non-classical theorems proved for the new constrained function can be applied for functions containing arbitrary number of free variables through functional instantiations with pseudo-lambda expressions.

The trick is to view the extra variable in the constrained function as a list of free variables $\Rightarrow$ no free variable appears in the functional instantiation.

Demo.

FTC-2 Proof Procedure

FTC-2: If f' is a real-valued continuous function on $[a, b]$ and f is an antiderivative of f' on $[a, b]$, then

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- Prove that f' returns real values on $[a, b]$.
- Prove that f' is continuous on $[a, b]$.
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FTC-2

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$$\int_a^b f'(x)dx = f(b) - f(a)$$

```
(defthm ftc-2
  (implies (and (inside-interval-p a (rcdfn-domain))
                (inside-interval-p b (rcdfn-domain)))
           (equal (int-rcdfn-prime a b)
                  (- (rcdfn b) (rcdfn a)))))
```

FTC-2

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$$\int_a^b f'(x)dx = f(b) - f(a)$$

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(defthm ftc-2
  (implies (and (inside-interval-p a (rcdfn-domain))
                (inside-interval-p b (rcdfn-domain)))
           (equal (int-rcdfn-prime a b)
                  (- (rcdfn b) (rcdfn a)))))
```

When functionally instantiating classical theorems, free variables are allowed to appear in pseudo-lambda expressions as long as classicalness is preserved [R. Gamboa & J. Cowles, 2007] $\Rightarrow$ use the "encapsulate trick" with zero-arity functions representing free variables.

Encapsulate Trick

Step 1: Define an encapsulate event that introduces zero-arity classical functions representing free variables.

Step 2: Prove that the zero-arity functions return standard values (use defthm-std).

Step 3: Prove the main theorem but replacing the free variables with corresponding zero-arity functions introduced in step 1. Without free variables, the functional instantiation can be applied straightforwardly.

Step 4: Prove the main theorem by functionally instantiating the zero-arity functions in the lemma proved in step 3 with free variables.

Outline

Fourier Coefficients

Theorem 4 (Fourier coefficients)

Let

$$f(x) = a_0 + \sum_{n=1}^{N}(a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$$

Then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\frac{\pi}{L}x)dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\frac{\pi}{L}x)dx$$

Sum Rule for Definite Integral of Indexed Sums

Lemma 5 (Sum rule for definite integral of indexed sums)

Let $\{f_n\}$ *be a set of real-valued continuous functions on* $[a, b]$, *where* $n = 0, 1, 2, ..., N$. *Then*

$$\int_a^b \sum_{n=0}^{N} f_n(x)dx = \sum_{n=0}^{N} \int_a^b f_n(x)dx$$

Sum Rule for Definite Integral of Indexed Sums

Proof. Let F_n be an antiderivative of f_n on $[a, b]$, where $n = 0, 1, 2, ..., N$. Then $\sum_{n=0}^{N} F_n(x)$ is an antiderivative of $\sum_{n=0}^{N} f_n(x)$ for all $x \in [a, b]$ by the sum rule for differentiation. By FTC-2, we have

$$\begin{aligned}\int_a^b \sum_{n=0}^{N} f_n(x)dx &= \sum_{n=0}^{N} F_n(b) - \sum_{n=0}^{N} F_n(a) \\ &= \sum_{n=0}^{N}(F_n(b) - F_n(a)) \\ &= \sum_{n=0}^{N} \int_a^b f_n(x)dx\end{aligned}$$

□

Fourier Coefficients

Theorem 4 (Fourier coefficients)

Let

$$f(x) = a_0 + \sum_{n=1}^{N}(a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$$

Then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\frac{\pi}{L}x)dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\frac{\pi}{L}x)dx$$

Fourier Coefficients

Proof. For $0 < m \leq N$,

$$
\begin{aligned}
&\frac{1}{L}\int_{-L}^{L} f(x)\cos(m\frac{\pi}{L}x)dx \\
=&\frac{1}{L}\int_{-L}^{L} \Big(a_0\cos(m\frac{\pi}{L}x) \\
&+\sum_{n=1}^{N}(a_n\cos(n\frac{\pi}{L}x)\cos(m\frac{\pi}{L}x)+b_n\sin(n\frac{\pi}{L}x)\cos(m\frac{\pi}{L}x))\Big)dx \\
=&\frac{1}{L}\Big(\int_{-L}^{L} a_0\cos(0\frac{\pi}{L}x)\cos(m\frac{\pi}{L}x)dx \\
&+\sum_{n=1}^{N}(\int_{-L}^{L} a_n\cos(n\frac{\pi}{L}x)\cos(m\frac{\pi}{L}x)dx+\int_{-L}^{L} b_n\sin(n\frac{\pi}{L}x)\cos(m\frac{\pi}{L}x)dx)\Big) \\
=&a_m
\end{aligned}
$$

Fourier Coefficients

Similarly, we have

$$\frac{1}{L}\int_{-L}^{L} f(x)\sin(m\frac{\pi}{L}x)dx = b_m,$$

$$\frac{1}{2L}\int_{-L}^{L} f(x)dx = a_0$$

□

Uniqueness of Fourier Sums

Corollary 6 (Uniqueness of Fourier sums)

Let

$$f(x) = a_0 + \sum_{n=1}^{N}(a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$$

and

$$g(x) = A_0 + \sum_{n=1}^{N}(A_n \cos(n\frac{\pi}{L}x) + B_n \sin(n\frac{\pi}{L}x))$$

$$\text{Then } f = g \Leftrightarrow \begin{cases} a_0 = A_0 \\ a_n = A_n, \text{ for all } n = 1, 2, ..., N \\ b_n = B_n, \text{ for all } n = 1, 2, ..., N \end{cases}$$

Uniquesness of Fourier Sums

Proof.

($\Rightarrow$) Follow immediately from the Fourier coefficient formula.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{L} g(x)dx = A_0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\frac{\pi}{L}x)dx = \frac{1}{L}\int_{-L}^{L} g(x)\cos(n\frac{\pi}{L}x)dx = A_n$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\frac{\pi}{L}x)dx = \frac{1}{L}\int_{-L}^{L} g(x)\sin(n\frac{\pi}{L}x)dx = B_n$$

($\Leftarrow$) Obviously true by induction on n.

□

Inner Product Formula

Theorem 7 (Inner product formula)

Let

$$f(x) = a_0 + \sum_{n=1}^{M} (a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$$

and

$$g(x) = A_0 + \sum_{n=1}^{N} (A_n \cos(n\frac{\pi}{L}x) + B_n \sin(n\frac{\pi}{L}x))$$

Then

$$\frac{1}{L}\int_{-L}^{L} f(x)g(x)dx = 2a_0 A_0 + \sum_{n=1}^{\min\{M,N\}} a_n A_n + \sum_{n=1}^{\min\{M,N\}} b_n B_n$$

Outline

Infinite Series

An infinite series is defined as the limit of the sequence of partial sums if the limit exists,

$$\sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=0}^{N} a_n$$

In non-standard analysis, it is defined as the standard part (st) of a partial sum with an infinitely large index H_0 if the sum is limited,

$$\sum_{n=0}^{\infty} a_n = \text{st}\left(\sum_{n=0}^{H_0} a_n\right)$$

where the natural number H_0 is i-large.

Definite Integral of an Infinite Series

Let's consider the following equality

$$\int_a^b \mathrm{st}(\sum_{n=0}^{H_0} f_n(x))dx \stackrel{?}{=} \mathrm{st}(\sum_{n=0}^{H_0} \int_a^b f_n(x)dx)$$

We can't claim it is true in general. However, we can prove that it is true under some conditions.

Pointwise Convergence vs. Uniform Convergence

Pointwise convergence: Suppose $\{f_n\}$ is a sequence of functions sharing the same domain and codomain. The sequence $\{f_n\}$ converges pointwise to f means that $f_H(x) \approx f(x)$ for all standard x in the domain of f_n and for all infinitely large natural numbers H.

Uniform convergence: Suppose $\{f_n\}$ is a sequence of functions sharing the same domain and codomain. The sequence $\{f_n\}$ converges uniformly to f means that $f_H(x) \approx f(x)$ for all x (both standard and non-standard) in the domain of f_n and for all infinitely large natural numbers H.

Assumptions

Assumption 1: $\{f_n\}$ is a sequence of real-valued continuous functions defined on a closed and bounded interval $[a, b]$.

Assumption 2: $f_n(x) \geq 0, \forall x \in [a, b]$ and $\forall n \in \mathbb{N}$.

Assumption 3: $\sum_{n=0}^{N} f_n(x)$ is limited $\forall x \in [a, b]$ and $\forall N \in \mathbb{N}$.

Assumption 4: $\exists c \in [a, b]$ s.t. $f_n(x) \leq f_n(c), \forall x \in [a, b]$ and $\forall n \in \mathbb{N}$.

Assumption 5: Let $g_N(x) = \sum_{n=0}^{N} f_n(x)$. Then the sequence $\{g_N\}$ is assumed to converge uniformly to $\text{st}(g_{H_0}(x))$ for some i-large $H_0 \in \mathbb{N}$, i.e., $\sum_{n=0}^{H} f_n(x) \approx \text{st}(\sum_{n=0}^{H_0} f_n(x))$, for all $x \in [a, b]$ and for all i-large $H \in \mathbb{N}$.

Definite Integral of an Infinite Series

Proof idea.

$$\int_a^b \mathrm{st}\left(\sum_{n=0}^{H_0} f_n(x)\right)dx \stackrel{?}{=} \mathrm{st}\left(\sum_{n=0}^{H_0} \int_a^b f_n(x)dx\right)$$

Proof idea.

$$\int_a^b \mathrm{st}(\sum_{n=0}^{H_0} f_n(x))dx \stackrel{?}{=} \mathrm{st}(\sum_{n=0}^{H_0} \int_a^b f_n(x)dx)$$

$$\int_a^b \mathrm{st}(\sum_{n=0}^{H_0} f_n(x))dx \stackrel{?}{=} \mathrm{st}(\int_a^b \sum_{n=0}^{H_0} f_n(x)dx) = \mathrm{st}(\sum_{n=0}^{H_0} \int_a^b f_n(x)dx)$$

From Assumptions 2 and 4, $\forall^{st} x \in [a, b]$

$$0 \le \sum_{n=0}^{H_0} f_n(x) - \sum_{n=0}^{N} f_n(x) = \sum_{n=N+1}^{H_0} f_n(x) \le \sum_{n=N+1}^{H_0} f_n(c)$$

$$\Rightarrow 0 \le \mathrm{st}(\sum_{n=0}^{H_0} f_n(x) - \sum_{n=0}^{N} f_n(x)) \le \mathrm{st}(\sum_{n=N+1}^{H_0} f_n(c))$$

$$\Rightarrow 0 \le \mathrm{st}(\sum_{n=0}^{H_0} f_n(x)) - \sum_{n=0}^{N} f_n(x) \le \mathrm{st}(\sum_{n=N+1}^{H_0} f_n(c)) \quad (1)$$

Definite Integral of an Infinite Series

By the transfer principle, (1) holds for all $x \in [a, b]$. Then, from Theorem 2, $\forall x \in [a, b]$

$$0 \le \int_a^b (\mathrm{st}(\sum_{n=0}^{H_0} f_n(x)) - \sum_{n=0}^{N} f_n(x))dx \le \mathrm{st}(\sum_{n=N+1}^{H_0} f_n(c))(b-a) \qquad (2)$$

From Assumption 5, $\sum_{n=0}^{H} f_n(x) \approx \mathrm{st}(\sum_{n=0}^{H_0} f_n(x)), \forall x \in [a, b]$

$$\Rightarrow \mathrm{st}(\sum_{n=0}^{H_0} f_n(x)) - \sum_{n=0}^{H} f_n(x) \approx 0, \forall x \in [a, b]$$

$$\Rightarrow \mathrm{st}(\sum_{n=H+1}^{H_0} f_n(x)) \approx 0, \forall x \in [a, b]$$

$$\Rightarrow \mathrm{st}(\sum_{n=H+1}^{H_0} f_n(c)) \approx 0, \text{since } c \in [a, b] \qquad (3)$$

Definite Integral of an Infinite Series

From (2) and (3), choose $N = H_0$

$$\int_a^b (\mathrm{st}(\sum_{n=0}^{H_0} f_n(x)) - \sum_{n=0}^{H_0} f_n(x))dx \approx 0$$

Next step

$$\int_a^b (\mathrm{st}(\sum_{n=0}^{H_0} f_n(x)) - \sum_{n=0}^{H_0} f_n(x))dx = \int_a^b \mathrm{st}(\sum_{n=0}^{H_0} f_n(x))dx - \int_a^b \sum_{n=0}^{H_0} f_n(x)dx$$

Then

$$\int_a^b \mathrm{st}(\sum_{n=0}^{H_0} f_n(x))dx \approx \int_a^b \sum_{n=0}^{H_0} f_n(x)dx$$

Or

$$\int_a^b \mathrm{st}(\sum_{n=0}^{H_0} f_n(x))dx = \mathrm{st}(\int_a^b \sum_{n=0}^{H_0} f_n(x)dx)$$

□

Assumptions

Assumption 1: $\{f_n\}$ is a sequence of real-valued continuous functions defined on a closed and bounded interval $[a, b]$.

Assumption 2: $f_n(x) \geq 0, \forall x \in [a, b]$ and $\forall n \in \mathbb{N}$.

Assumption 3: $\sum_{n=0}^{N} f_n(x)$ is limited $\forall x \in [a, b]$ and $\forall N \in \mathbb{N}$.

Assumption 4: $\exists c \in [a, b]$ s.t. $f_n(x) \leq f_n(c), \forall x \in [a, b]$ and $\forall n \in \mathbb{N}$.

Assumption 5: Let $g_N(x) = \sum_{n=0}^{N} f_n(x)$. Then the sequence $\{g_N\}$ is assumed to converge uniformly to $\mathrm{st}(g_{H_0}(x))$ for some i-large $H_0 \in \mathbb{N}$, i.e., $\sum_{n=0}^{H} f_n(x) \approx \mathrm{st}(\sum_{n=0}^{H_0} f_n(x))$, for all $x \in [a, b]$ and for all i-large $H \in \mathbb{N}$.

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Dini Uniform Convergence Theorem

Theorem 8 (Dini uniform convergence theorem)

A monotone sequence of continuous functions $\{f_n\}$ that converges pointwise to a continuous function f on a closed and bounded interval $[a, b]$ is uniformly convergent.

Dini Uniform Convergence Theorem

Proof idea. Without loss of generality, assume $\{f_n\}$ is monotonically increasing. $\forall x \in [a, b], \forall$ i-large $H \in \mathbb{N}$

$$
\begin{aligned}
&|f_H(x) - f(x)| \\
=&|f_H(x) - f_H(\text{st}(x)) + f_H(\text{st}(x)) - f(\text{st}(x)) + f(\text{st}(x)) - f(x)| \\
\leq&|f_H(x) - f_H(\text{st}(x))| + |f_H(\text{st}(x)) - f(\text{st}(x))| + |f(\text{st}(x)) - f(x)|
\end{aligned}
$$

Lemma: If $x \in [a, b]$ then $\text{st}(x) \in [a, b]$ (note that this is only true on closed and bounded intervals).

Since $\text{st}(x)$ is standard, $f_H(\text{st}(x)) \approx f(\text{st}(x))$ by the pointwise convergence of $\{f_n\}$.

Since $\text{st}(x)$ is standard and $x \approx \text{st}(x)$, $f(\text{st}(x)) \approx f(x)$ by the continuity of f.

If we can show that $f_H(x) \approx f_H(\text{st}(x))$, then $f_H(x) \approx f(x)$ for all $x \in [a, b]$.

Dini Uniform Convergence Theorem

By the continuity of $\{f_n\}$, we have $f_n(x) \approx f_n(\mathrm{st}(x)), \forall x \in [a, b]$ and $\forall^{st} n \in \mathbb{N}$.

Proof obligation: $f_H(x) \approx f_H(\mathrm{st}(x)), \forall x \in [a, b]$ and $\forall H \in \mathbb{N}$.

Idea: Apply the overspill principle in non-standard analysis.

Overspill Principle

Overspill principle: Let $P(n, x)$ be a classical predicate. Then

$$\forall x(\forall^{st} n \in \mathbb{N}, P(n, x) \Rightarrow \exists^{\neg st} k \in \mathbb{N}, P(k, x))$$

Apply the above principle, we can even come up with a stronger statement as follows:

Let $P(n, x)$ be a classical predicate. Then

$$\forall x(\forall^{st} n \in \mathbb{N}, P(n, x) \Rightarrow \exists^{\neg st} k \in \mathbb{N}, \forall m \in \mathbb{N}(m \leq k \Rightarrow P(m, x)))$$

Dini Uniform Convergence Theorem

Define a classical predicate $P(n, x, x_0)$ as follows:

$$P(n, x, x_0) \equiv |f_n(x) - f_n(x_0)| < \frac{1}{n+1}$$

Let $x \in [a, b]$ and $x_0 = \text{st}(x)$, then $P(n, x, \text{st}(x))$ holds for all standard $n \in \mathbb{N}$ by the continuity of $\{f_n\}$. Hence, by the overspill principle, there exists a non-standard $k \in \mathbb{N}$ s.t. $P(m, x, \text{st}(x))$ holds for all $m \in \mathbb{N}$ and $m \leq k$.

If m is non-standard, then

$$0 \leq |f_m(x) - f_m(\text{st}(x))| < \frac{1}{m+1} \approx 0$$

$$\Rightarrow f_m(x) \approx f_m(\text{st}(x))$$

Thus, $f_H(x) \approx f(x)$ for all $x \in [a, b]$ and for all i-large $H \leq k$.

Dini Uniform Convergence Theorem

If $H > k$, then by the monotonicity of $\{f_n\}$

$$0 \leq |f_H(x) - f(x)| \leq |f_k(x) - f(x)| \approx 0$$

$$\Rightarrow f_H(x) \approx f(x), \forall x \in [a, b]$$

□

Relaxing Assumption 5

Assumption 5: Let $g_N(x) = \sum_{n=0}^{N} f_n(x)$. Then the sequence $\{g_N\}$ is assumed to converge uniformly to $\text{st}(g_{H_0}(x))$ for some i-large $H_0 \in \mathbb{N}$, i.e., $\sum_{n=0}^{H} f_n(x) \approx \text{st}(\sum_{n=0}^{H_0} f_n(x))$, for all $x \in [a, b]$ and for all i-large $H \in \mathbb{N}$.

Relaxing Assumption 5

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Assumption 5.1: The limit function $\text{st}(\sum_{n=0}^{H_0} f_n(x))$ is continuous on $[a, b]$ for some i-large $H_0 \in \mathbb{N}$.

Assumption 5.2: Let $g_N(x) = \sum_{n=0}^{N} f_n(x)$. Then the sequence $\{g_N\}$ is assumed to converge pointwise to the continuous function $\text{st}(g_{H_0}(x))$ for some i-large $H_0 \in \mathbb{N}$, i.e., $\sum_{n=0}^{H} f_n(x) \approx \text{st}(\sum_{n=0}^{H_0} f_n(x)), \forall^{st} x \in [a, b]$ and for all i-large $H \in \mathbb{N}$.

Assumptions

Assumption 1: $\{f_n\}$ is a sequence of real-valued continuous functions defined on a closed and bounded interval $[a, b]$.

Assumption 2: $f_n(x) \geq 0, \forall x \in [a, b]$ and $\forall n \in \mathbb{N}$.

Assumption 3: $\sum_{n=0}^{N} f_n(x)$ is limited $\forall x \in [a, b]$ and $\forall N \in \mathbb{N}$.

Assumption 4: $\exists c \in [a, b]$ s.t. $f_n(x) \leq f_n(c), \forall x \in [a, b]$ and $\forall n \in \mathbb{N}$.

Assumption 5.1: The limit function $\text{st}(\sum_{n=0}^{H_0} f_n(x))$ is continuous on $[a, b]$ for some i-large $H_0 \in \mathbb{N}$.

Assumption 5.2: Let $g_N(x) = \sum_{n=0}^{N} f_n(x)$. Then the sequence $\{g_N\}$ is assumed to converge pointwise to the continuous function $\text{st}(g_{H_0}(x))$ for some i-large $H_0 \in \mathbb{N}$, i.e., $\sum_{n=0}^{H} f_n(x) \approx \text{st}(\sum_{n=0}^{H_0} f_n(x)), \forall^{st} x \in [a, b]$ and for all i-large $H \in \mathbb{N}$.

Outline

Conclusions and Future Work

Adding an extra argument in constrained functions solves the problem of functionally instantiating non-classical theorems with classical functions containing free variables.

Free variables are allowed to appear in pseudo-lambda expressions when functionally instantiating classical theorems, as long as classicalness is preserved.

Fourier coefficient formula can be formalized in ACL2(r) as described.

Still remain a couple of proof obligations in formalizing the definite integral of an infinite series. E.g., the overspill principle in proving Dini uniform convergence theorem.

It would be nice if we can apply ACL2(r) to circuit verification!

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Questions!