Expressing Symmetry Breaking in Propositional Proofs

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Joint work with
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Motivation

Satisfiability solvers are used in amazing ways...

- Hardware verification: Centaur x86 verification
- Combinatorial problems:
  - Ramsey numbers and van der Waerden numbers [Dransfield, Marek, and Truszczynski, 2004; Kouril and Paul, 2008]
  - Gardens of Eden in Conway's Game of Life [Hartman, Heule, Kwekkeboom, and Noels, 2013]
  - Erdős Discrepancy Problem [Konev and Lisitsa, 2014]

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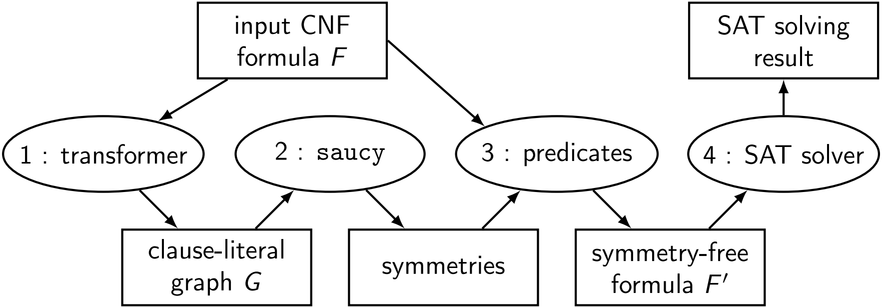
..., but satisfiability solvers have errors.

- Documented bugs in SAT, SMT, and QBF solvers [Brummayer and Biere, 2009; Brummayer et al., 2010]
- Implementation errors often imply conceptual errors
- Symmetry breaking, which is crucial to solve combinatorial problems, cannot be validated with existing methods

Symmetry Breaking Tool Chain

1. The input formula is transformed into a clause-literal graph;
2. A symmetry detection tool extracts symmetries from the graph;
3. Symmetry-breaking predicates are added to the input formula;
4. The symmetry-free formula is solved using a SAT solver.



A bug in any of these tools may result in incorrect results

Most observed bugs during SAT Competition 2013 were caused by tools 1-3

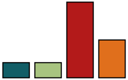
From Resolution to Clausal DRAT Proofs

Easy to Emit

Compact

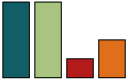
Checked Efficiently

Expressive



Resolution Proofs

Zhang and Malik, 2003
Van Gelder, 2008; Biere, 2008



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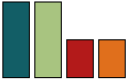
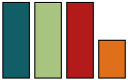


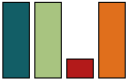
Clausal proofs + clause deletion

Heule, Hunt, Jr., and Wetzler [STVR 2014]



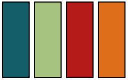
Optimized clausal proof checker

Heule, Hunt, Jr., and Wetzler [FMCAD 2013]



Clausal RAT proofs

Heule, Hunt, Jr., and Wetzler [CADE 2013]



DRAT proofs (RAT + deletion)

Wetzler, Heule, and Hunt, Jr. [SAT 2014]

From Resolution to Clausal DRAT Proofs

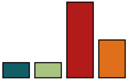
Easy to Emit

Compact

Checked Efficiently

Expressive

Verified



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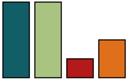

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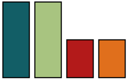

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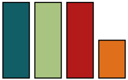

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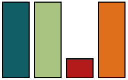

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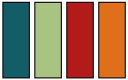


DRAT proofs (RAT + deletion)
Wetzler, Heule, and Hunt, Jr. [SAT 2014]

Main Contribution

We present a method to express the addition of symmetry-breaking predicates in DRAT, a clausal proof format supported by top-tier solvers.

Our method allows, for the first time, validation of SAT solver results obtained via symmetry breaking, thereby validating the results of symmetry extraction tools as well.

Symmetry Breaking in SAT Solvers

Solution Symmetry

A truth assignment τ is a set of non-complementary literals. τ satisfies formula F if it contains a literal for each clause in F.

A signed variable permutation $\pi := (x_1, \ldots, x_n)(p_1, \ldots, p_n)$ maps literals x_i onto p_i and $\bar{x}_i$ onto $\bar{p}_i$ with p_i either equal to x_j or $\bar{x}_j$ and $\text{var}(p_i) \neq \text{var}(p_j)$ if $i \neq j$ with $1 \leq i, j \leq n$.

Example

- Let $\tau = \{x, \bar{y}, z\}$ and $\pi = (x, y, z)(y, \bar{z}, \bar{x})$.
- $\pi(\tau) = \{y, z, \bar{x}\}$

$T(F)$: the set of satisfying truth assignments for formula F.

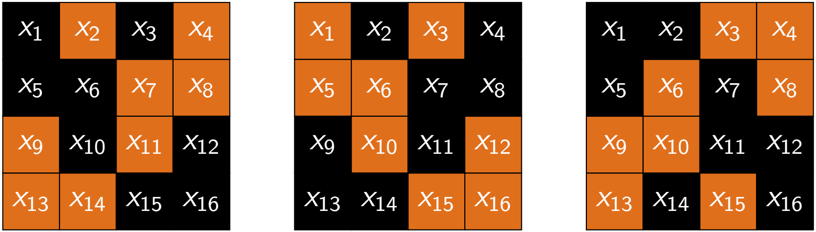
A solution symmetry σ for a given formula F is a signed variable permutation such that $\forall \tau . \tau \in T(F) \rightarrow \sigma(\tau) \in T(F)$

Solution Symmetries and Non-Monochromatic Rectangles

A solution symmetry σ for a given formula F is a signed variable permutation such that $\forall \tau . \tau \in T(F) \rightarrow \sigma(\tau) \in T(F)$

Color the cells of a 4x4 grid either orange (0) or black (1) such that all rectangles have non-monochromatic corners.

This problem contains many solutions symmetries, for example $\sigma_1 = (x_1, \ldots, x_{16})(\bar{x}_1, \ldots \bar{x}_{16})$, $\sigma_2 = (x_1, \ldots, x_{16})(x_{16}, \ldots, x_1)$



τ $\sigma_1(\tau)$ $\sigma_2(\tau)$

Breaking a Solution Symmetry

A solution symmetry σ for a given formula F is a signed variable permutation such that $\forall \tau . \tau \in T(F) \rightarrow \sigma(\tau) \in T(F)$

Let $\sigma = (x_1, \ldots, x_n)(p_1, \ldots, p_n)$. A symmetry-breaking predicate B_σ is the constraint $x_1, \ldots, x_n \leq p_1, \ldots, p_n$.

Example (using $\sigma = (x_1, x_2)(x_2, x_1)$)

- $B_\sigma : x_1, x_2 \leq x_2, x_1$, which blocks $\tau = \{x_1, \bar{x}_2\}$.
- in clausal form: $(\bar{x}_1 \vee x_2)$.

Symmetry breaking: if F contains symmetry σ, add B_σ to F.

If satisfiable formula F contains a symmetry σ then $F \not\models B_\sigma$.

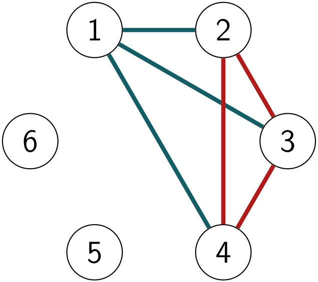
Central idea: transform F into an equi-satisfiable formula F' by replacing literals l with new literals l', such that $F' \models B'_\sigma$.

Example Formulas: Unavoidable Subgraphs

A connected undirected graph G is an unavoidable subgraph of clique K of order n if any red/blue edge-coloring of the edges of K contains G either in red or in blue.

Ramsey Number $R(k)$: What is the smallest n such that any graph with n vertices has either a clique or a co-clique of size k?

$$\begin{aligned} & R(3) = 6 \\ & R(4) = 18 \\ 43 \leq\ & R(5) \leq 49 \end{aligned}$$



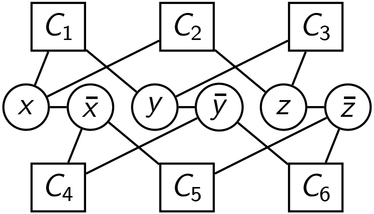
SAT solvers can determine that $R(4) = 18$ in 1 second using symmetry breaking; w/o symmetry breaking it requires weeks.

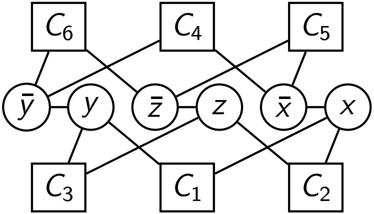
Example formula: an unavoidable path of two edges

Consider the formula below — which expresses the statement whether path of two edges unavoidable in a clique of order 3:

$$F := \overbrace{(x \vee y)}^{C_1} \wedge \overbrace{(x \vee z)}^{C_2} \wedge \overbrace{(y \vee z)}^{C_3} \wedge \overbrace{(\bar{x} \vee \bar{y})}^{C_4} \wedge \overbrace{(\bar{x} \vee \bar{z})}^{C_5} \wedge \overbrace{(\bar{y} \vee \bar{z})}^{C_6}$$

A clause-literal graph has a vertex for each clause and literal, and edges for each literal occurrence connecting the literal and clause vertex. Also, two complementary literals are connected.





Symmetry: $(x, y, z)(\bar{y}, \bar{z}, \bar{x})$ is an edge-preserving bijection

Convert Symmetries into Symmetry-Breaking Predicates

A symmetry $\sigma = (x_1, \ldots, x_n)(p_1, \ldots, p_n)$ of a CNF formula F is an edge-preserving bijection of the clause-literal graph of F, that maps literals x_i onto p_i and $\bar{x}_i$ onto $\bar{p}_i$ with $i \in \{1..n\}$.

Given a CNF formula F. Let τ be a satisfying truth assignment for F and σ a symmetry for F, then $\sigma(\tau)$ is also a satisfying truth assignment for F.

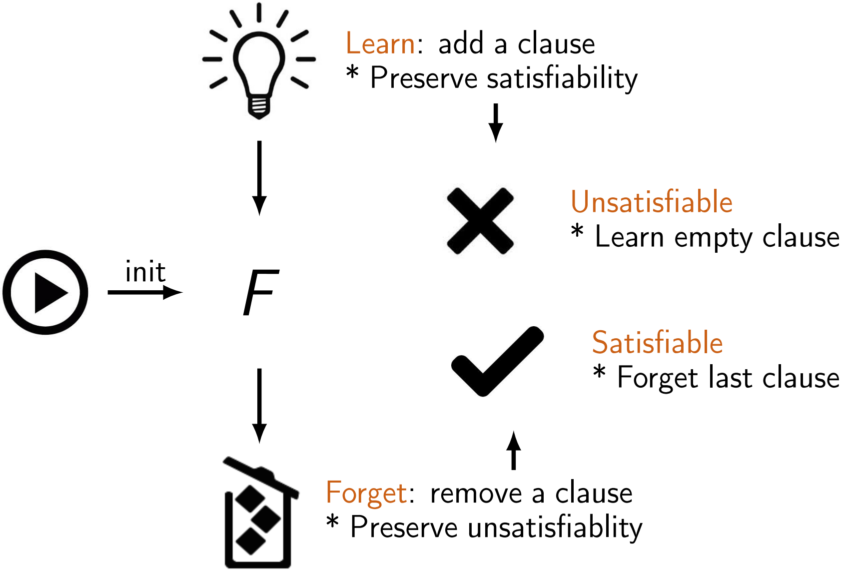
Symmetry $\sigma = (x_1, \ldots, x_n)(p_1, \ldots, p_n)$ for F can be broken by adding a symmetry-breaking predicate: $x_1, \ldots, x_n \leq p_1, \ldots, p_n$.

$$(\bar{x}_1 \vee p_1) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee p_2) \wedge (p_1 \vee \bar{x}_2 \vee p_2) \wedge$$
$$(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee p_3) \wedge (\bar{x}_1 \vee p_2 \vee \bar{x}_3 \vee p_3) \wedge$$
$$(p_1 \vee \bar{x}_2 \vee \bar{x}_3 \vee p_3) \wedge (p_1 \vee p_2 \vee \bar{x}_3 \vee p_3) \wedge \ldots$$

Why are we allowed to add these clauses?

Breaking a Single Symmetry

Clausal Proof System



Resolution Asymmetric Tautology (RAT) [IJCAR 2012]

Given a clause $C = (l_1 \vee \cdots \vee l_k)$ and a CNF formula F:

- $\overline{C}$ denotes the conjunction of its negated literals $(\bar{l}_1) \wedge \cdots \wedge (\bar{l}_k)$
- $F \vdash_1 \epsilon$ denotes that unit propagation on F derives a conflict
- C is an asymmetric tautology w.r.t. F if and only if $F \wedge \overline{C} \vdash_1 \epsilon$
- C is a resolution asymmetric tautology on $l \in C$ w.r.t. F iff for all resolvents $C \diamond D$ with $D \in F$ and $\bar{l} \in D$ holds that $F \wedge \overline{C \diamond D} \vdash_1 \epsilon$

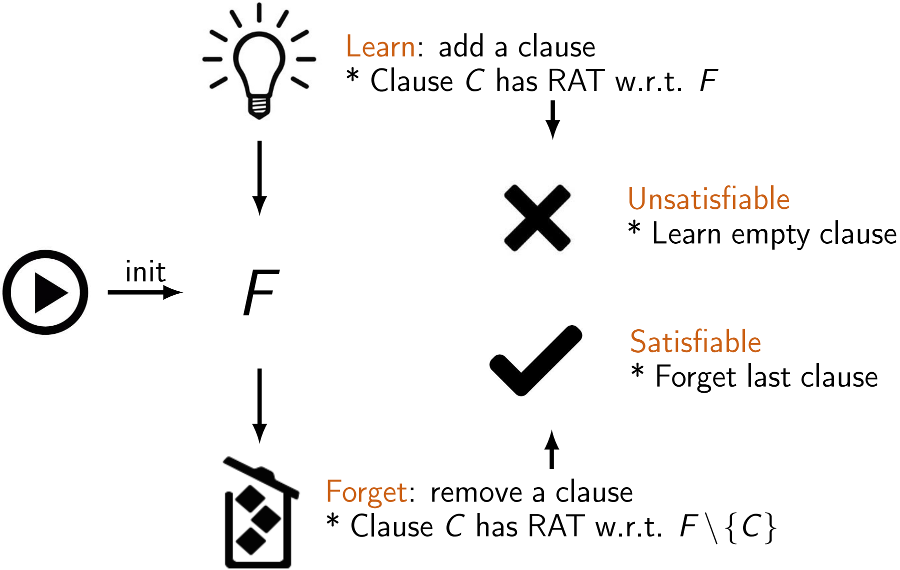
Example

Consider the formula $F = (a \vee c) \wedge (\bar{b} \vee \bar{c}) \wedge (b \vee d)$:

- The clause $(a \vee d)$ is an asymmetric tautology w.r.t. F
- The clause $(b \vee c)$ is an resolution asymmetric tautology w.r.t. F

Theorem: Given a formula F and a clause C having RAT with respect to F, then F and $F \cup \{C\}$ are equi-satisfiable.

Clausal Proof System using RAT addition and deletion



Expressing a Symmetry Breaking Predicate in DRAT (1)

Introduce auxiliary variables using $\sigma = (x_1, \dots, x_n)(p_1, \dots, p_n)$

- The swap variable $s := x_1, \dots, x_n > p_1, \dots, p_n$
- The prime variables $x'_i := \begin{cases} p_i & \text{if } s \text{ set to true} \\ x_i & \text{otherwise} \end{cases}$

Example (using $\sigma = (x_1, x_2)(x_2, x_1)$)

- $add(s \vee \bar{x}_1 \vee x_2), add(\bar{s} \vee x_1), add(\bar{s} \vee \bar{x}_2)$
- $add(x'_1 \vee \bar{s} \vee \bar{x}_2)$, $add(\bar{x}'_1 \vee \bar{s} \vee x_2)$, $add(x'_1 \vee s \vee \bar{x}_1)$, $add(\bar{x}'_1 \vee s \vee x_1)$

Add symmetry-breaking predicate using the prime variables:

- Add the constraint $x'_1, \dots, x'_n \leq p'_1, \dots, p'_n$

Example (using $\sigma = (x_1, x_2)(x_2, x_1)$)

- $add(s \vee \bar{x}'_1 \vee x'_2), add(\bar{x}'_1 \vee x'_2), delete(s \vee \bar{x}'_1 \vee x'_2)$

Expressing a Symmetry Breaking Predicate in DRAT (2)

Redefine involved clauses

- For each clause $C \in F$ that contains at least one literal l which occurs in the symmetry, add a clause C' which is a copy of C with literals l' for each such l.
- Remove the original involved clauses.

Example (using $\sigma = (x_1, x_2)(x_2, x_1)$ and $C = (x_2 \vee \bar{x}_3)$)

- $add(s \vee x_2' \vee \bar{x}_3)$, $add(x_2' \vee \bar{x}_3)$, $delete(s \vee x_2' \vee \bar{x}_3)$, $delete(x_2 \vee \bar{x}_3)$

Optionally remove all definitions of the first step

- After this step, the resulting formula is equal to the original formula extended with the symmetry-breaking predicate (modulo variable renaming).
- This step reduces validation costs significantly.

Breaking Multiple Symmetries

Difficulties due to Multiple Symmetries: Example

Consider the formula $F = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \vee (x_2 \vee x_3) \vee (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$ and its two symmetries: $\sigma_1 = (x_1, x_2)(x_2, x_1)$ and $\sigma_2 = (x_2, x_3)(x_3, x_2)$.

Symmetry breaking will add the predicates $(\bar{x}_1 \vee x_2)$ and $(\bar{x}_2 \vee x_3)$. Hence the number of predicates is linear in the number of symmetries.

Breaking σ_1 using the method shown results in the formula
$F' = (x'_1 \vee x'_2) \wedge (x'_1 \vee x_3) \vee (x'_2 \vee x_3) \vee (\bar{x}'_1 \vee \bar{x}'_2 \vee \bar{x}_3) \wedge (\bar{x}'_1 \vee x'_2)$

Breaking σ_2 afterwards results in a problem as the predicate $(\bar{x}'_1 \vee x'_2)$ cannot be redefined: σ_2 is not a symmetry for F'.

The method shown could only redefine original clauses. As a consequence, to break both σ_1 and σ_2, the method shown needs to be applied multiple times: σ_1, σ_2, and again σ_1.

Difficulties due to Multiple Symmetries: General

Consider a formula F with two symmetries
$\sigma_1 = (x_1, \ldots, x_n)(p_1, \ldots, p_n)$ and
$\sigma_2 = (y_1, \ldots, y_n)(q_1, \ldots, q_n)$.

Symmetry breaking will add the predicates
$x_1, \ldots, x_n \leq p_1, \ldots, p_n$ and $y_1, \ldots, y_n \leq q_1, \ldots, q_n$. The number of predicates is linear in the number of symmetries.

After breaking σ_1 with the method shown, resulting in formula F', we cannot simply apply it again, because σ_2 is not a symmetry of F'.

Hence, the method shown can only redefine original clauses.

To obtain a symmetry-free formula, the method shown has to be applied multiple times per symmetry. Even with just two symmetries, one may needs to apply it twice per symmetry.

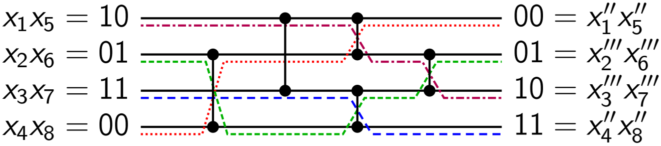
Break a Symmetry Chain using Sorting Networks

A symmetry chain is a sequence of k symmetries of length $2n$ with the property that $x_{i,j} = p_{i,j+n}$, $p_{i,j} = x_{i,j+n}$, and $x_{i+1,j} = x_{i,j+n}$ with $1 \leq i < k$ and $1 \leq j \leq n$.

Example (symmetry chain $(x_1, x_5)(x_2, x_6)(x_3, x_7)(x_4, x_8)$)

- based on $\sigma_i = (x_i, x_{i+4}, x_{i+1}, x_{i+5})(x_{i+1}, x_{i+5}, x_i, x_{i+4})$
- results in predicates $x_1, x_5 \leq x_2, x_6 \leq x_3, x_7 \leq x_4, x_8$

Breaking a symmetry chain comes down at sorting the assignments, which can be realized using a sorting network.



Size k symmetry chain: apply the procedure $\mathcal{O}(k \log^2 k)$ times.

Converting Symmetries into a Symmetry Chain

Q: How to break multiple symmetries in general?

A: Convert them into a symmetry chain.

$+$: Limits the size of the partial proof.

$-$: Breaks the symmetries only partially.

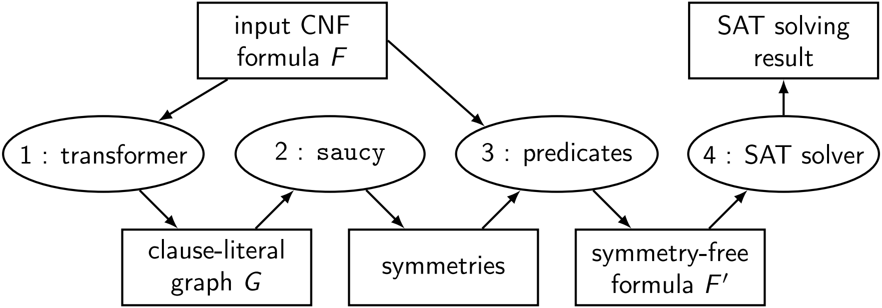
Example

Consider two symmetries: $\sigma_1 = (x_1, x_4, x_2, x_5)(x_2, x_5, x_1, x_4)$ and $\sigma_2 = (x_2, x_4, x_3, x_6)(x_3, x_6, x_2, x_4)$. Compute reduced symmetries $\sigma_1' = (x_1, x_2)(x_2, x_1)$ and $\sigma_2' = (x_2, x_3)(x_3, x_2)$ that form a symmetry chain. Using σ_1' and σ_2' to define the swap variable and the symmetry-breaking predicate. Use σ_1 and σ_2 for the other definitions.

Tools and Evaluation

Old Tool Chain

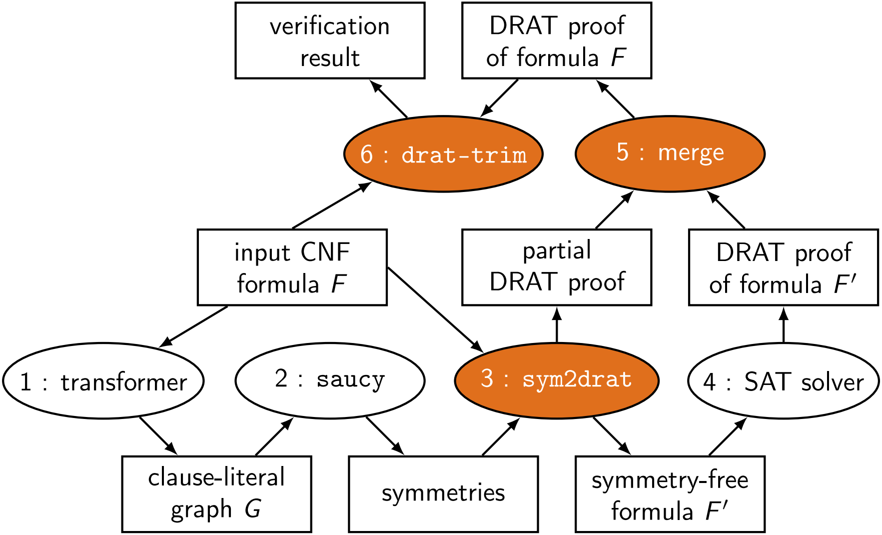
1. The input formula is transformed into a clause-literal graph;
2. A symmetry detection tool extracts symmetries from the graph;
3. Symmetry-breaking predicates are added to the input formula;
4. The symmetry-free formula is solved using a SAT solver.



A bug in any of these tools may result in incorrect results

Most observed bugs during SAT Competition 2013 were caused by tools 1-3

New Tool Chain



Only the correctness of the proof checker needs to be trusted

New Tools

The new tool `sym2drat`:

- Input: CNF formula and symmetries;
- Output: A symmetry-free formula and a partial proof that describes the derivation from the input formula to the symmetry-free formula;
- Uses pairwise sorting to reduce the size of the partial proof.

Merge: simply concatenate using Unix `cat`

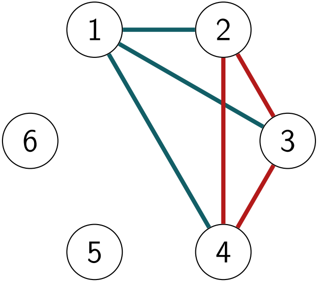
DRAT proof checkers:

- Implemented an extension for `drat-trim` to validate partial DRAT proofs. This feature was crucial during development for debugging purposes;
- Modified our mechanically-verified proof checker to make it compatible with DRAT proofs.

Evaluation: Ramsey Number Four

Ramsey Number $R(k)$: What is the smallest n such that any graph with n vertices has either a clique or a co-clique of size k?

$$\begin{aligned} R(3) &= 6 \\ R(4) &= 18 \\ 43 \leq \quad R(5) &\leq 49 \end{aligned}$$

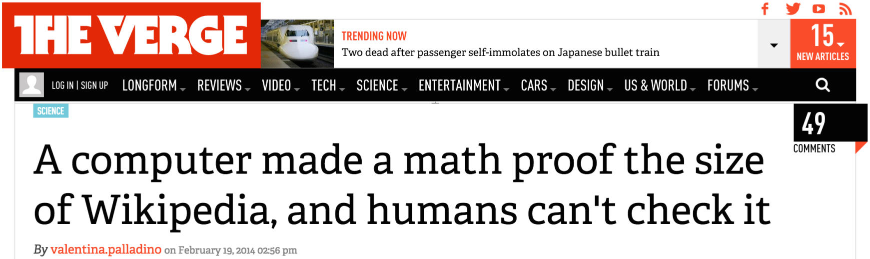


SAT solvers can determine that $R(4) = 18$ in 1 second using symmetry breaking; w/o symmetry breaking it requires weeks.

The size of the proof is 20 MB and the time required is 1.8 s

Proof validated with our mechanically-verified checker as well.

Evaluation: Erdős Discrepancy Conjecture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By valentina.palladino on February 19, 2014 02:56 pm

Erdős Discrepancy Conjecture was recently solved using SAT.

The conjecture states that there exists no infinite sequence of -1, +1 such that for all d, k holds that ($x_i \in \{-1, +1\}$):

$$\left|\sum_{i=1}^{k} x_{id}\right| \leq 2$$

The original DRAT proof was 13Gb. Our new proof using symmetry breaking is 2Gb.

Evaluation: Two Pigeons per Hole Problems

Biere proposed benchmarks expressing whether $2n+1$ pigeons can be put in n holes that contain at most two pigeons per hole.

For $n > 6$ they can only be solved by symmetry breaking or cardinality resolution.

No SAT solvers can produce a proof for the problems with $n > 6$.

Our method can produce proofs for problems with $n \leq 12$ that can be generated in minutes and validated within an hour.

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- The first approach to validate symmetry-breaking techniques usage in SAT solvers by expressing the techniques as DRAT proof steps;
- Increases the trust in results based on symmetry breaking;
- Evaluated our method on hard-combinatorial formulas.

Future work:

- Determine precisely the number of times the symmetry-breaking procedure needs to be applied;
- Improve the speed of the mechanically-verified checker;
- Implement a parallel proof checker to reduce the gap between solving and verification costs.

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Thanks! Questions?