PRuning Through Satisfaction

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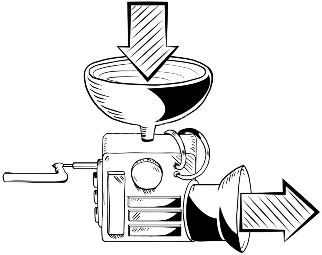
Satisfiability Solving (Highly Simplified)

SAT problem:
Given a propositional formula, is it satisfiable?

$(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$



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Input Formula in CNF

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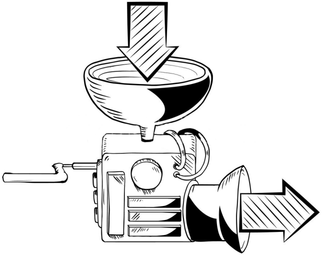

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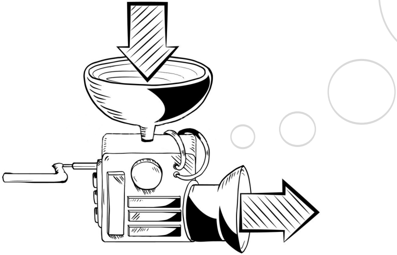

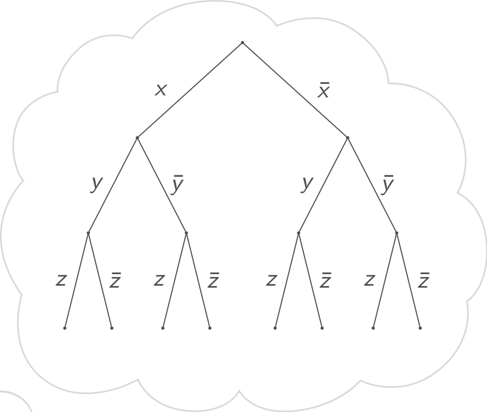

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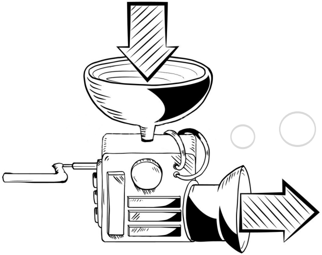

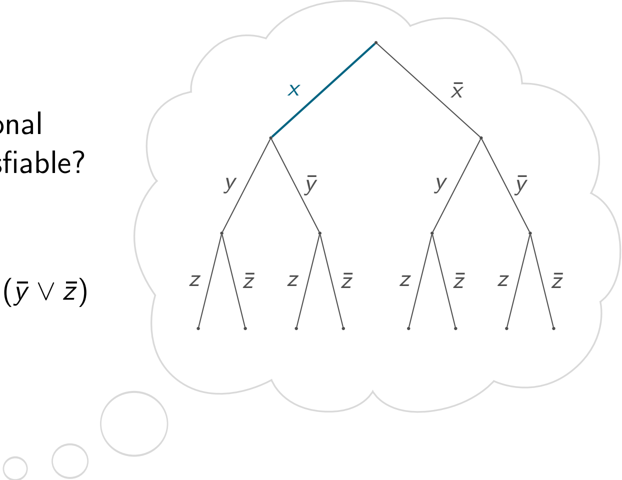

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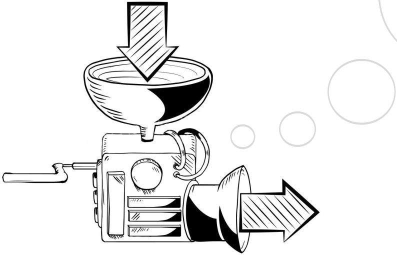

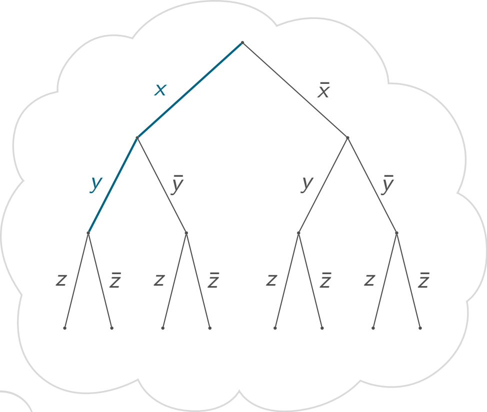

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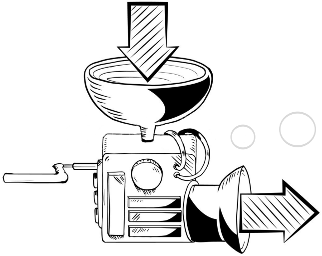

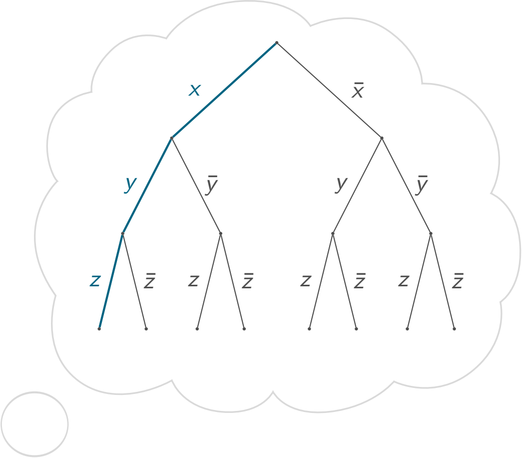

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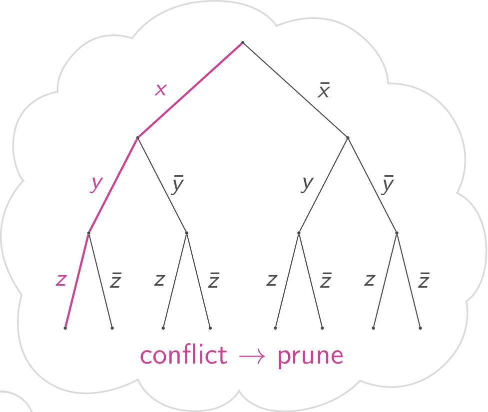

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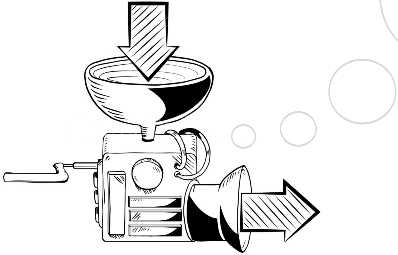

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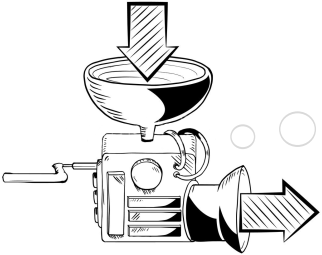

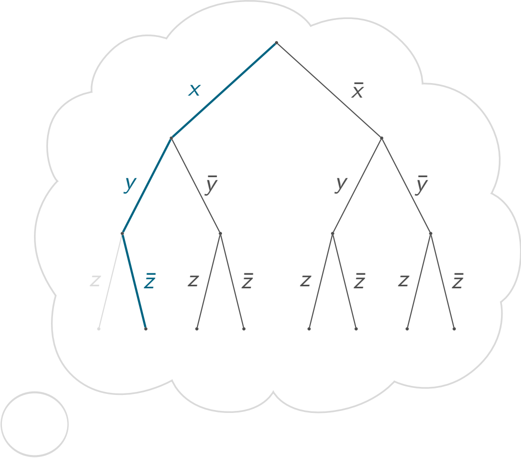

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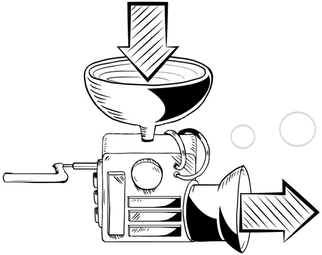
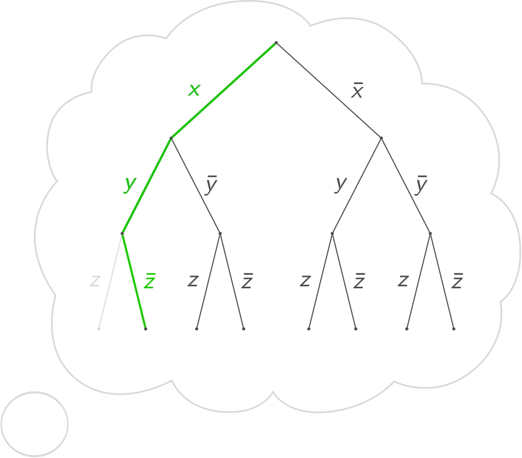
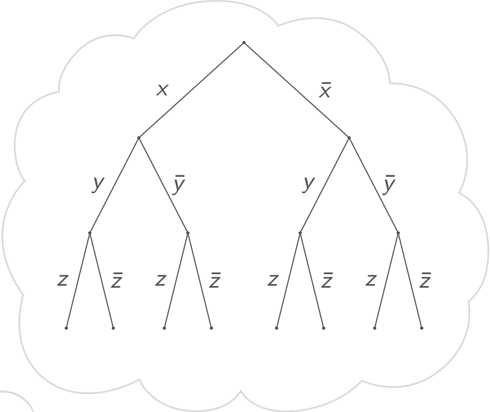




Satisfiable

Key Idea: Prune Less Satisfiable Branches

Can we prune earlier?
Even satisfiable branches?



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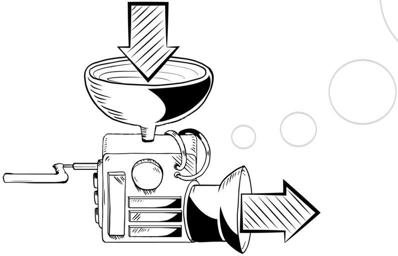

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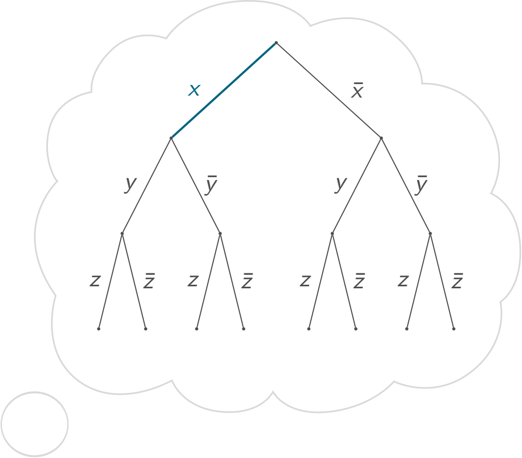

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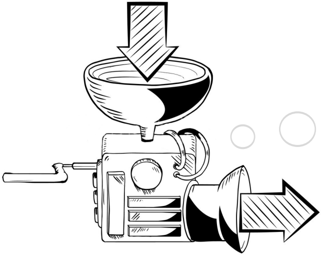

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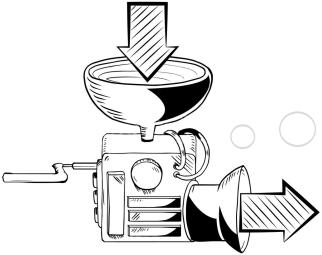

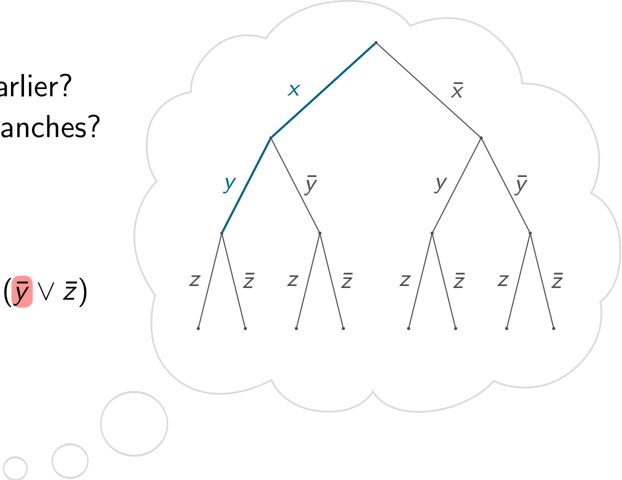

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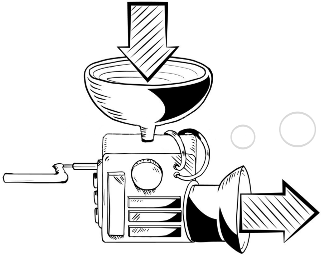

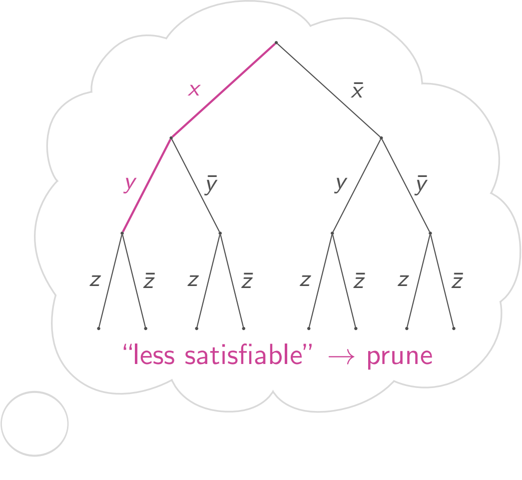

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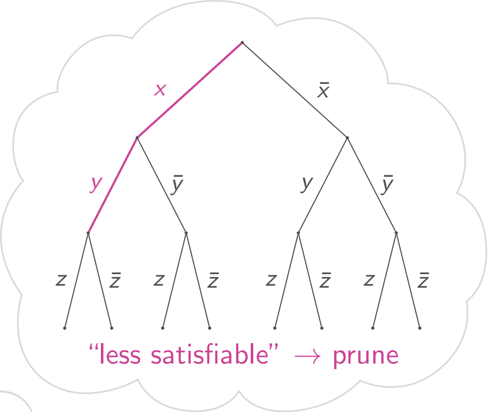

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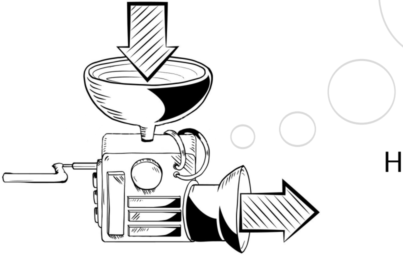
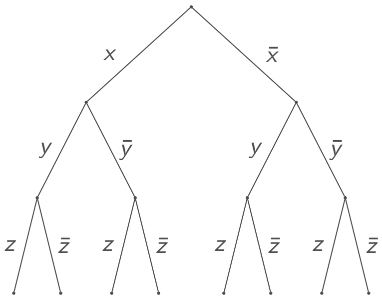


How to prune? Add redundant clauses!

Pruning via Clause Addition

- A clause prunes all branches that falsify the clause.



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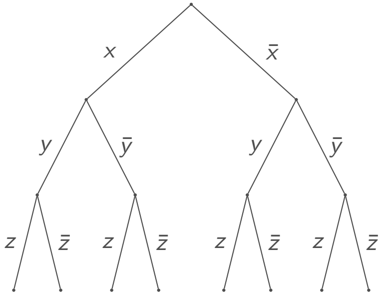

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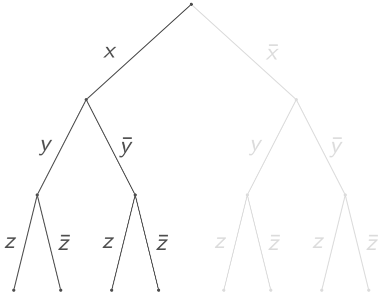

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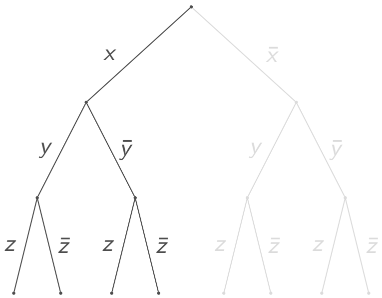

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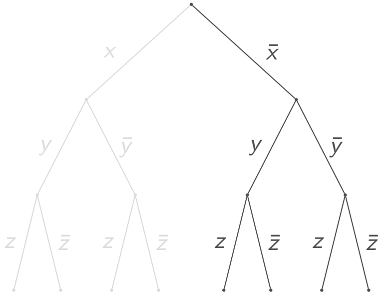

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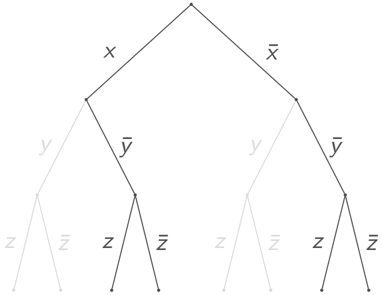

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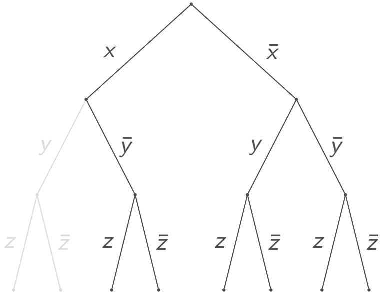

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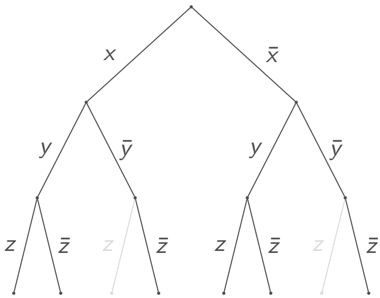

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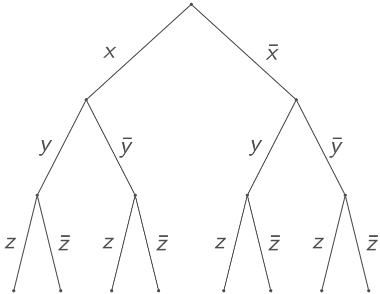

The Positive Reduct

Traditional Proofs vs. Interference-Based Proofs

- In traditional proof systems, everything that is inferred, is logically implied by the premises.

$$\frac{C \vee x \qquad \bar{x} \vee D}{C \vee D}\,(\text{res}) \qquad\qquad \frac{A \qquad A \rightarrow B}{B}\,(\text{mp})$$

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➥ Inference rules reason about the presence of facts.

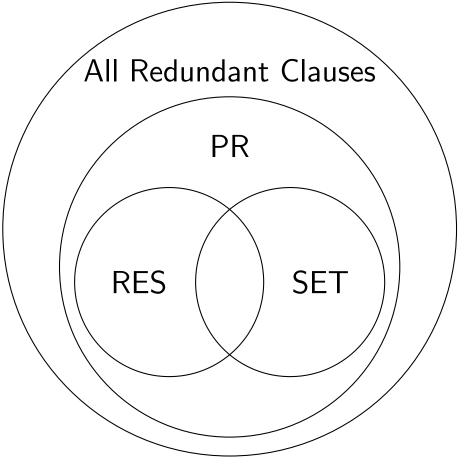
- If certain premises are present, infer the conclusion.

- Different approach: Allow not only implied conclusions.
  - Require only that the addition of facts preserves satisfiability.
  - Reason also about the absence of facts.

  ➥ This leads to interference-based proof systems.

Redundant Clauses

A clause C is redundant w.r.t. a formula F if and only if F and $F \wedge C$ are either both satisfiable or both unsatisfiable.



PR = Propagation Redundant Clauses [CADE'17]

RES = Resolvents

SET = Set-Blocked Clauses [IJCAR'16]

Finding Redundant Clauses: The Positive Reduct

Determining whether a clause C is SET or PR w.r.t. a formula F is an NP-complete problem.

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Positive reducts are typically very easy to solve!

Key Idea: While solving a formula F, check whether the positive reduct of F and the current assignment α is satisfiable. In that case, prune the branch α.

The Positive Reduct: An Example

Given a formula F and a clause C. Let α denote the smallest assignment that falsifies C. The positive reduct of F and α, denoted by $p(F, \alpha)$, is the formula that contains C and all *assigned*(D, α) with $D \in F$ and D is satisfied by α.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

Let $C_1 = (\bar{x})$, so $\alpha_1 = x$.
The positive reduct $p(F, \alpha_1) = (\bar{x}) \wedge (x) \wedge (x)$ is unsatisfiable.

Let $C_2 = (\bar{x} \vee \bar{y})$, so $\alpha_2 = x\, y$.
The positive reduct $p(F, \alpha_2) = (\bar{x} \vee \bar{y}) \wedge (x \vee y) \wedge (x \vee \bar{y})$ is satisfiable.

Conditional Autarkies

Autarkies

A non-empty assignment α is an autarky for formula F if every clause $C \in F$ that is touched by α is also satisfied by α.

A pure literal and a satisfying assignment are autarkies.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.
Assignment $\alpha_1 = \bar{z}$ is an autarky: $(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.
Assignment $\alpha_2 = x\,\bar{y}\,z$ is an autarky: $(x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.

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Given an assignment α, $F|_\alpha$ denotes a formula F without the clauses satisfied by α and without the literals falsified by α.

Theorem ([Monien and Speckenmeyer 1985])

Let α be an autarky for formula F.
Then, F and $F|_\alpha$ are satisfiability equivalent.

Conditional Autarkies

An assignment $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ is a conditional autarky for formula F if α_{aut} is an autarky for $F|_{\alpha_{\text{con}}}$.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{z})$.
Let $\alpha_{\text{con}} = x$ and $\alpha_{\text{aut}} = \bar{y}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\,\bar{y}$ is a conditional autarky for F:

$$\alpha_{\text{aut}} = \bar{y} \text{ is an autarky for } F|_{\alpha_{\text{con}}} = (\bar{y} \vee \bar{z}).$$

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Let $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ be a conditional autarky for formula F. Then F and $F \wedge (\alpha_{\text{con}} \rightarrow \alpha_{\text{aut}})$ are satisfiability-equivalent.

In the above example, we could therefore learn $(\bar{x} \vee \bar{y})$.

Learning PR clauses

Theorem

Given a formula F and an assignment α. Every satisfying assignment ω of $p(F, \alpha)$ is a conditional autarky of F.

Recall: Given a formula F and a clause C. Let α denote the smallest assignment that falsifies C. C is SET w.r.t. F if and only if $p(F, \alpha)$ is satisfiable.

Let assignment ω satisfy $p(F, \alpha)$. Removing all but one of the literals in C that are satisfied by ω results in a PR clause w.r.t. F.

The Algorithm

Pseudo-Code of CDCL (formula F)

$\alpha := \emptyset$
forever do
    $\alpha :=$ Simplify (F, α)
    if $F|_\alpha$ contains a falsified clause **then**
        $C :=$ AnalyzeConflict $()$
        if C is the empty clause **then return** unsatisfiable
        $F := F \cup \{C\}$
        $\alpha :=$ BackJump (C, α)
    else
        $l :=$ Decide $()$
        if l is undefined **then return** satisfiable
        $\alpha := \alpha \cup \{l\}$

Pseudo-Code of SDCL (formula F)

```
α := ∅
forever do
    α := Simplify (F, α)
    if F|α contains a falsified clause then
        C := AnalyzeConflict ()
        if C is the empty clause then return unsatisfiable
        F := F ∪ {C}
        α := BackJump (C, α)
    else if p(F, α) is satisfiable then
        C := AnalyzeWitness ()
        F := F ∪ {C}
        α := BackJump (C, α)
    else
        l := Decide ()
        if l is undefined then return satisfiable
        α := α ∪ {l}
```

Evaluation

Benchmark Suite: Pigeon Hole Formulas

Can $n{+}1$ pigeons be placed in n holes (at-most-one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \le p \le n+1} (x_{1,p} \vee \cdots \vee x_{n,p}) \wedge \bigwedge_{1 \le h \le n,\, 1 \le p < q \le n+1} (\overline{x}_{h,p} \vee \overline{x}_{h,q})$$

The binary clauses encode the constraint $\le_1 (x_{h,1}; \ldots; x_{h,n+1})$.

There exists more compact encodings, such as the sequential counter and minimal encoding, for at-most-one constraints.

We include these encodings to evaluate the robustness of the solver.

Tool Comparison

We used three tools in our evaluation:

- EBDDRES: A tool based on binary decision diagrams that can convert a refutation into an extended resolution proof.
- GLUCOSER: A SAT solver with extended learning, i.e., a technique that introduces new variables and could potentially solve pigeon hole formulas in polynomial time.
- LINGELING (PR): Our SDCL solver.

Results on Small Pigeon Hole Formulas

	input		EBDDRES		GLUCOSER		LINGELING (PR)	
formula	#var	#cls	time	#node	time	#lemma	time	#lemma
PHP_{10}-std	110	561	1.00	3M	22.71	329,470	0.07	329
PHP_{11}-std	132	738	3.47	9M	146.61	1,514,845	0.11	439
PHP_{12}-std	156	949	10.64	27M	307.29	2,660,358	0.16	571
PHP_{13}-std	182	1,197	30.81	76M	982.84	6,969,736	0.22	727
PHP_{10}-seq	220	311	OF	——	1.62	25,712	0.07	327
PHP_{11}-seq	264	375	OF	——	6.94	77,747	0.10	437
PHP_{12}-seq	312	445	OF	——	19.40	174,084	0.14	569
PHP_{13}-seq	364	521	OF	——	172.76	1,061,318	0.18	725
PHP_{10}-min	180	281	28.60	81M	0.64	15,777	0.06	329
PHP_{11}-min	220	342	143.92	399M	1.82	34,561	0.10	439
PHP_{12}-min	264	409	OF	——	9.87	121,321	0.13	571
PHP_{13}-min	312	482	OF	——	57.66	483,789	0.18	727

OF = 32-bit overflow

Results on Large Pigeon Hole Formulas

	input		EBDDRES		GLUCOSER		LINGELING (PR)	
formula	#var	#cls	time	#node	time	#lemma	time	#lemma
PHP_{20}-std	420	4,221	OF	——	TO	——	1.61	2,659
PHP_{30}-std	930	13,981	OF	——	TO	——	13.45	8,989
PHP_{40}-std	1,640	32,841	OF	——	TO	——	67.41	21,319
PHP_{50}-std	2,550	63,801	OF	——	TO	——	241.14	41,649
PHP_{20}-seq	840	1,221	OF	——	TO	——	1.05	2,657
PHP_{30}-seq	1,860	2,731	OF	——	TO	——	6.55	8,987
PHP_{40}-seq	3,280	4,841	OF	——	TO	——	27.10	21,317
PHP_{50}-seq	5,100	7,551	OF	——	TO	——	86.30	41,647
PHP_{20}-min	760	1,161	OF	——	TO	——	1.03	2,659
PHP_{30}-min	1,740	2,641	OF	——	TO	——	6.30	8,989
PHP_{40}-min	3,120	4,721	OF	——	TO	——	26.65	21,319
PHP_{50}-min	4,900	7,401	OF	——	TO	——	85.00	41,649

OF = 32-bit overflow TO = timeout of 9000 seconds

Conclusions and Future Work

Conclusions

SDCL generalizes the well-known CDCL paradigm by allowing to prune branches that are potentially satisfiable:

- Such branches can be found using the positive reduct;
- Pruning can be expressed in the PR proof system;
- Runtime and proofs can be exponentially smaller.

Our SDCL solver finds short proofs of pigeon hole formulas:

- Integrated in the state-of-the-art solver Lingeling;
- Linear sized proofs in $\mathcal{O}(n^3)$ can be found fully automatically;
- The implementation is efficient, robust, and open source.

Future Work

- SDCL likely requires different heuristics compared to CDCL
- Can more branches be pruned using stronger SAT calls?
- How to minimize clauses from pruning through satisfaction?
- Can SLS techniques be used to find conditional autarkies?

PRuning Through Satisfaction

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