An Integration of Axiomatic Set Theory with ACL2

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UT Austin (retired)

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Outline

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Books use no trust tags and required **no ACL2 changes**.

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Key new insight last Fall: ACL2 can be a pure set-theory prover by encoding ACL2 primitives and data into set theory.

Additional motivation: Provides a vehicle for embedding higher-order logic (HOL) developments into ACL2.

- That could be the subject of future talks.

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- Other ACL2 objects are encoded as discussed later, e.g.:
  - `Cons` is represented using the Kuratowski ordered pair:
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Let's look at this picture from Wikipedia:
$V_{\omega * \omega}$

EXAMPLES

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- Classical set theory example: Cantor's theorem
  (Let's look briefly at the certifiable book, `cantor.lisp`);
  we'll revisit it later after providing more background.
- "Higher-order function" example: `map`
  - We'll look at `(defun map ...)` in `base.lisp` and the two theorems following it. First note:
    - In `(map f lst)`, think of `f` as a set of ordered pairs and `lst` as an ACL2 list.
    - I'll explain later how `(defthmz ... :props ...)` can be viewed as `(defthm ...)`.
  - We'll look at `zify.lisp` to see an application of `map` to the Fibonacci function.
  - Not discussed here: See `foldr.lisp`.

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Goal: Provide a platform for efficient set-theory reasoning.

- The axioms need justification, but need not be minimal.
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Let's look at the exports in the first `encapsulate` form in `base.lisp`, up to "Embedding of ACL2 data types".

- Notice the local witness of `nil` for `zfc`, which serves as a hypothesis!
- A metatheoretic argument provides a meaningful interpretation for which `(zfc)` is true.
- Not included there: Comprehension (Subset) or Replacement (equivalently, Collection) schemes of ZF (to be discussed later)

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Review of First Talk

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The initial `encapsulate` event in `base.lisp` introduces *hypothesis function* `zfc` and primitives `in`, `pair`, `min-in`, `union`, `omega`, and `powerset`, along with `subset` and some basic axioms.

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But for now, let's return to the initial `encapsulate` event in `base.lisp` and see how ACL2 atoms and consing are defined in set theory.

ENCODING ACL2 OBJECTS IN SET THEORY

Let's look at the rest of that initial `encapsulate` in `base.lisp` to see how ACL2 data type recognizers are defined — and also at the definitions of `relation-p` and `funp` after that.

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TIP: Note, as in `funp`, the use of `non-exec` in `defun-sk` to support `guard` verification.

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Comprehension in ZF

The *Comprehension* (or *Subset*) scheme of ZF says that the intersection of a predicate with a set is a set.

- Informally: $\{a \in x : P(a)\}$ is a set.
- Formal statement, for each formula P with y not free:
  $\forall x \exists y \forall a (a \in y \Leftrightarrow (a \in x \wedge P))$

ZSUB EXAMPLE

From `base.lisp`:

```
; The following defines the Cartesian product
; (prod2 a b)
; as:
; {p \in (powerset (powerset (union2 a b))) :
;  (prod-member p a b)}

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Let's see how this call of `zsub` expands, using `:trans1` and focusing on `PROD2$COMPREHENSION`.

MORE ZSUB EXAMPLES

As time permits we'll take a quick look at more examples in `base.lisp`:

`domain, inverse, codomain, compose`

ZFC-TABLE

Recall that the `prod2` example above generates:

```
(TABLE ZFC-TABLE
  'PROD2$PROP
  '(ZSUB PROD2 (A B)
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Key property: Every key of `zfc-table` is a zero-ary function symbol that returns true in our underlying set theory.

Thus: The `table` guard of `zfc-table` checks that `prod2$prop` can be assumed to hold by the Comprehension scheme.

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  - After all, adding a bunch of `T` hypotheses has no logical effect.
- The default value for `:props` is `(zfc)`.
- `Defthmdz` and `thmz` similarly extend `defthmd` and `thm` (respectively) with a `:props` argument.

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`Make-event` tips from
`:trans1 (CHECK-PROPS DEFTHMZ (ZFC PROD2$PROP)):`

- **TIP**: Use `:expansion?` to avoid bloat in `.cert` file.
- **TIP**: Use `:on-behalf-of :quiet` to suppress noisy output
- **TIP**: Use `:check-expansion t` to ensure that the check is made even at `include-book` time.

SIMPLIFYING EXPORTS FROM ZSUB

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Let's look at the proof of `in-prod2`, which simplifies `prod2$comprehension`.

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ORDINALS

Let's look at these key events in the section "Omega is an ordinal" in `base.lisp` (with "`:guard t`" omitted).

```
(defun-sk in-is-linear (s)
  (forall (x y) (implies (and (in x s)
                              (in y s)
                              (not (equal x y)))
                         (or (in x y)
                             (in y x)))))
(defun-sk transitive (x)
  (forall a (implies (in a x)
                     (subset a x)))
  :rewrite :direct)
(defun ordinal-p (x)
  (and (in-is-linear x)
       (transitive x)))
(defthmz ordinal-p-omega (ordinal-p (omega)))
```

ORDINALS (CONTINUED)

See `ordinals.lisp` for more theorems about ordinals.
(Again, this is work in progress.)
A key result:

```
(defthmz ordinal-trichotomy
  (implies (and (ordinal-p a)
                (ordinal-p b)
                (not (in a b))
                (not (in b a)))
           (equal (equal a b)
                  t))
  :props (zfc diff$prop)
  :hints ...)
```

Iterated Composition

See `iterate.lisp`.

De Morgan's Laws Etc.

See `set-algebra.lisp`.

Function Spaces

See `fun-space.lisp`.

Reasoning about Free Variables and Quantifiers

Example: see `demo1.lsp` for a proof of the theorem `domain-union2` from `set-algebra.lisp`.

- **TIP**: Enable `extensionality-rewrite` to prove two sets are equal.
- **TIP**: Let forcing help you to find missing `:props`.
- **TIP**: Use the proof-builder for generalization and for rewriting possibilities; also see :DOC `:pl`.
- **TIP**: Replace proof-builder rewrites involving free variables by `:restrict` hints.

Transfinite Induction

Time permitting, we'll talk about *epsilon-induction* and look at the macro `prove-inductive-suffices` and the examples below it in `induction.lisp`.

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Transfinite induction on the ordinals is a special case of epsilon-induction.

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ACL2's version allows F to be a **relation** whose domain **need not contain** A, and concludes that there is a **set-theoretic function** — a set of ordered pairs — based on the restriction of F to A.

- That's a mouthful — we'll look at the picture again and I'll illustrate with an example on the next slide.
- The ACL2 version follows easily from the ZF axioms.

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(zfn v ()                            ; name, args
     x y                             ; x, y
     (omega)                         ; bound for x
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(defun v-omega ()
  (declare (xargs :guard t))
  (union (codomain (v))))
```

GOOD ACL2 OBJECTS ARE IN V_ω

Recognizer for "good ACL2 object":

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(defun acl2p (x)
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- `:trans* t (prove-acl2p mirror)`
- **TIP**: Use `:trans*` instead of `:trans1` when `make-event` is involved.

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Transitive Closure

We'll skip this slide unless there is time for it.

A set is *transitive* if every member of a member is a member, i.e., every member is a subset.

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(defun-sk transitive (x)
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Time permitting, we'll look at theorems labeled "A key theorem" in `tc.lisp`.

Perhaps we'll also look at the definition of `tc` in file `tc.lisp`.

```
(defun tc-n (n s) ...)
(zfn tc-fn (s) ...)
(defun tc (s) ...)
```

Outline

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ZIFY Introduction: Revisiting FIB

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:trans1 (zify zfib fib :dom (omega) :ran (omega))
```

Below is a key part of the zify call above, informally:
$\{\langle p_1, p_2 \rangle \in \omega \times \omega : p_2 = \texttt{fib}(p_1)\}$.

```
(zsub zfib ()
      p
      (prod2 (omega) (omega))
      (equal (cdr p) (fib (car p))))
```

ZIFY EXAMPLE: MIRROR

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Fix it using `thmz` (and show `zify-prop`):

```
(thmz (equal (apply (zmirror) '((a . b) . (c . d)))
             '((d . c) . (b . a)))
      :props (zify-prop acl2$prop v$prop zmirror$prop))
```

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`Zify*` is a variant of `zify` that can convert arbitrary-arity ACL2 functions to set-theoretic functions.

- The idea is to get a unary function that maps arglists to values.

See `zify.lisp` for a few examples.
(I haven't used `zify*` much.)

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See `cantor.lisp` for a straightforward adaptation of the formalization and proof on Wikipedia.

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- Note the natural use of `zsub` to follow the Wikipedia proof.
- **TIP**: Note the use of `minimal-theory` for control of the proof.
- **TIP**: It's OK to leave proof-builder `:instructions` when they're easily maintainable.

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- **TIP**: Hand proofs can be helpful; see `schroeder-bernstein-main-2-2` in `schroeder-bernstein-support.lisp`.

Outline

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Future Work (Highly Incomplete List!)

- Transfinite recursion, e.g., V_α for all ordinals α
- Cardinals, cardinality (in progress)
- Higher-order applications (e.g., temporal logics)
- Tool improvements, e.g., let `zsub` return `:REDUNDANT`
- More automation
  - ACL2 modification for parity-based rewriting (or maybe use existing clause-processor?)
  - Quantifier instantiation (maybe Dave Greve's stuff?)
  - Automated functional instantiation (use existing work?)
- Prove correctness for the embedding of ACL2 into ZFG.
- More set theory
  - ω_1 (soon; should be easy using Cantor's theorem)
  - Cofinality, closed unbounded subsets, stationary sets
  - Mostowski collapse
  - Independence results
  - Basic topology
  - ...

Wrapping Up

Thank you for your attention!

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Possible PhD dissertation topic(s)?
Collaborators?