CS 357 Assignment #3

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There are two parts to this assignment, which are described in Sections 1 and 2, respectively.

1 Programming & Problem Solving

This part of the assignment consists of a number of exercises related to matroids. The concept of a matroid is introduced in Section 1.1. The exercises are stated in Section 1.2; solutions are due at the beginning of class on Wednesday, October 5. Students are allowed to work on this part of the assignment with a partner, and are strongly encouraged to do so. Each team should turn in only one set of solutions. If you are having trouble finding a partner, send me an email and I will match you up with someone else in the same situation.

Section 1.3 describes the matroid greedy algorithm. In the next assignment, we will use the matroid greedy algorithm to compute an MWMCM of a certain restricted kind of CBG.

1.1 Matroids

A matroid is an ordered pair $(S, \mathcal{I})$, where S is a finite nonempty set, and $\mathcal{I}$ is a nonempty collection of subsets of S, called the *independent* subsets of S, such that the following two properties hold: (1) if A belongs to $\mathcal{I}$ and $B \subseteq A$, then B belongs to $\mathcal{I}$ (hereditary property); (2) if A and B belong to $\mathcal{I}$ and $|A| > |B|$, then there is an x in $A \setminus B$ such that $B \cup \{x\}$ belongs to $\mathcal{I}$ (exchange property).

Here is a concrete example of a matroid, called a *graphic matroid*. Fix an arbitrary undirected graph $G = (V, E)$. Define $\mathcal{I}$ as the set of all acyclic subsets of E, i.e., the set of all subsets A of E such that the graph (V, A) is acyclic. It is not too difficult to verify that $(E, \mathcal{I})$ is a matroid.

An independent set A of a matroid is *maximal* if it is not properly contained in any other independent set. In the case of a graphic matroid, the maximal independent sets correspond to the spanning forests of the graph G. (If G is connected, then the maximal independent sets correspond to the spanning trees of G.)

1.2 Exercises

Let $G = (V, E)$ be an undirected graph. Assume that V is partitioned into a set L of "left" vertices and a set R of "right" vertices, and that every edge in E connects a left vertex to a right vertex. (Thus, G is bipartite.) We say that a set of left vertices A is *feasible* if there is a matching of G that matches every vertex in A.

(a) Briefly indicate why the empty set of left vertices is feasible.

(b) Briefly indicate why any subset of a feasible set of left vertices is feasible.

(c) Prove that if A and B are two feasible sets of left vertices such that $|A| > |B|$, then there exists a left vertex u in $A \setminus B$ such that $B \cup \{u\}$ is feasible.

(d) Let $\mathcal{A}$ denote the set of all feasible sets of left vertices. Briefly indicate why $(L, \mathcal{A})$ is a matroid. Hint: Use the results of parts (a), (b), and (c).

(e) A feasible set of left vertices A is said to be *maximal* if it is not properly contained in any other feasible set of left vertices. Prove that every feasible set of left vertices is contained in some maximal feasible set of left vertices.

(f) Let k denote the size of a maximum cardinality matching (MCM) of G. Prove that every maximal feasible set of left vertices has cardinality k. Hint: Use the result of part (c).

(g) Prove that if A is a feasible set of left vertices, then there is an MCM M of G such that every vertex in A is matched in M. Hint: Use the results of parts (e) and (f).

1.3 The Matroid Greedy Algorithm

A *weighted* matroid is a matroid $(S, \mathcal{I})$ where each element of S has an associated weight. A fundamental matroid optimization problem is the following: Given a weighted matroid, determine a minimum-weight maximal independent set. (The weight of an independent set is defined as the sum of the weights of the elements of the set.)

In the special case of a graphic matroid, the problem of determining a minimum-weight maximal independent set corresponds to the minimum spanning forest problem. Recall that Kruskal's algorithm can be used to solve the minimum spanning forest problem. Can Kruskal's algorithm be generalized to solve the minimum-weight maximal independent set problem for arbitrary matroids? Yes, Kruskal's algorithm is a special case of the *matroid greedy algorithm*, which does just that.

The matroid greedy algorithm works as follows. First, we sort the elements of S in nondecreasing order of weight, breaking ties arbitrarily, and we initialize A to the empty set, which is guaranteed to be independent. (That the empty set is independent follows from the hereditary property, and the requirement that there is at least one independent set.) Then, we consider each x in S in turn, in the order established by the sort. When we consider x, we add it to A if and only if $A \cup \{x\}$ is independent. After considering all of the elements of S, we claim that the set A is a minimum-weight maximal independent set. The proof of this claim is similar to the proof of correctness of Kruskal's algorithm; we will not review it here.

The same framework can be used to determine a maximum-weight maximal independent set. The only difference is that we sort the elements of S in nonincreasing order, rather than in nondecreasing order.

2 Textbook Exercises

This part of the assignment is due at the beginning of class on Wednesday, October 12.

1. Problem 4.13, page 194.
2. Problem 4.22, page 200.
3. Problem 5.1, page 246.
4. Problem 5.7, page 248.