CS313H

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Challenge

- Define $A = \mathbb{R} - \{-1\}$, and define $f : A \to \mathbb{R}$ by $f(a) = 2a/(a+1)$.
  Prove f is injective but not surjective.

Good Morning, Colleagues

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Are there any questions?

Logistics

- Next week has relatively little new material
  - Time for concepts to sink in
  - Test review

Quiz!

- Write the **power set** of $\{A, 1\}$:
  $P(\{A, 1\}) = ?$

- Write the **Cartesian product** of $\{A, B\}$ and $\{C, D\}$:
  $\{A, B\} \times \{C, D\} = ?$

- Which of the pictures on the board is an **injection**?

- Which of the pictures on the board is a **surjection**?

- Which of the pictures on the board is a **bijection**?

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  3. Since $b \in B$, and f is onto, there is an $a \in A$ s.t. $f(a) = b$.

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  Prove f is injective (one-to-one) but not surjective (onto).

Prove or disprove

NOTE - 1st is on the homework, and something very related to the 2nd is on the homework. Try to do the 2nd on piazza

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Cantor-Bernstein-Schröder Theorem

- If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

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5. Let $f(x) = f(y)$ where $x, y \in \mathbb{N}$
6. Then $(a_{r_1}, q_1 + 1) = (a_{r_2}, q_2 + 1)$ where $(x-1) = q_1 n + r_1$ and $(y-1) = q_2 n + r_2$

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16. Therefore $f(x) = (a_i, m)$

Assignments for Tuesday

- Fourth homework **due at start of class**
- Modules 16.6 with associated readings