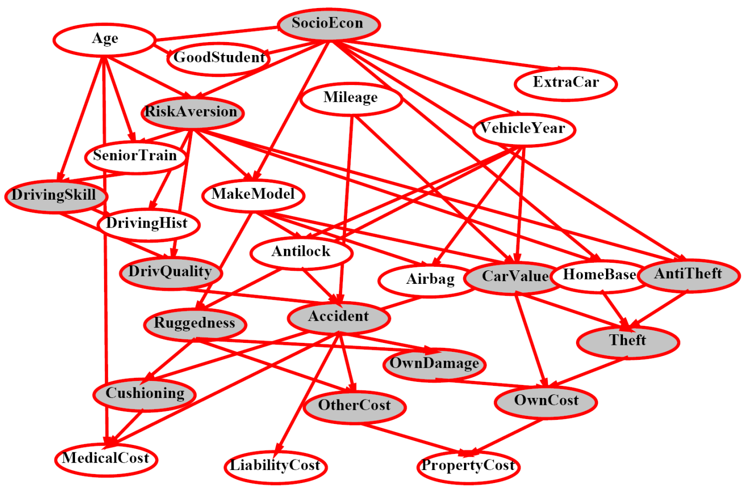
Bayes' Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain



- Questions we can ask:
  - Inference: given a fixed BN, what is P(X | e)?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

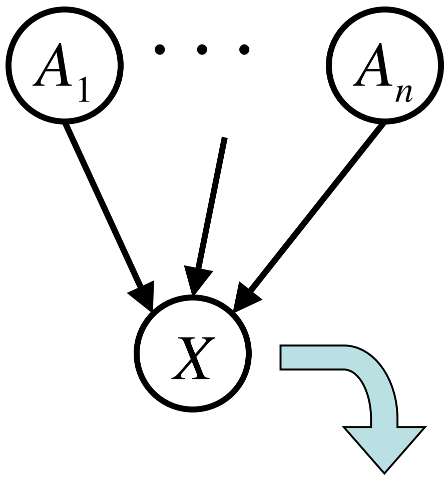
This slide deck courtesy of Dan Klein at UC Berkeley

Bayes' Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \ldots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process

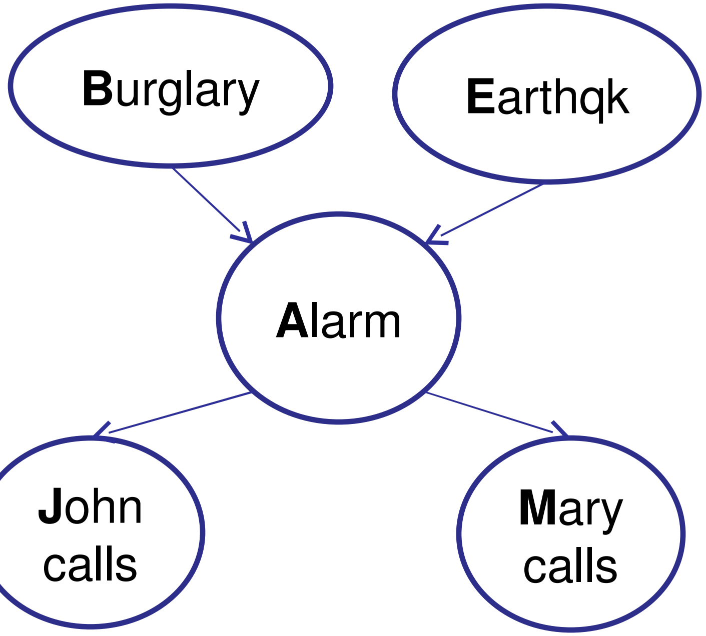


$$P(X|A_1 \ldots A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

Example: Alarm Network

B	P(B)
+b	0.001
¬b	0.999



E	P(E)
+e	0.002
¬e	0.998

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	¬a	0.05
+b	¬e	+a	0.94
+b	¬e	¬a	0.06
¬b	+e	+a	0.29
¬b	+e	¬a	0.71
¬b	¬e	+a	0.001
¬b	¬e	¬a	0.999

A	J	P(J|A)
+a	+j	0.9
+a	¬j	0.1
¬a	+j	0.05
¬a	¬j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	¬m	0.3
¬a	+m	0.01
¬a	¬m	0.99

Probabilities in BNs

- For all joint distributions, we have (chain rule):

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1, \ldots, x_{i-1})$$

- Bayes'nets *implicitly* encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

- This lets us reconstruct any entry of the full joint
- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

Same Assumptions, Different Graphs?

- Can you have two different graphs that encode the same assumptions?
  - Yes!
  - Examples:

Example: Independence

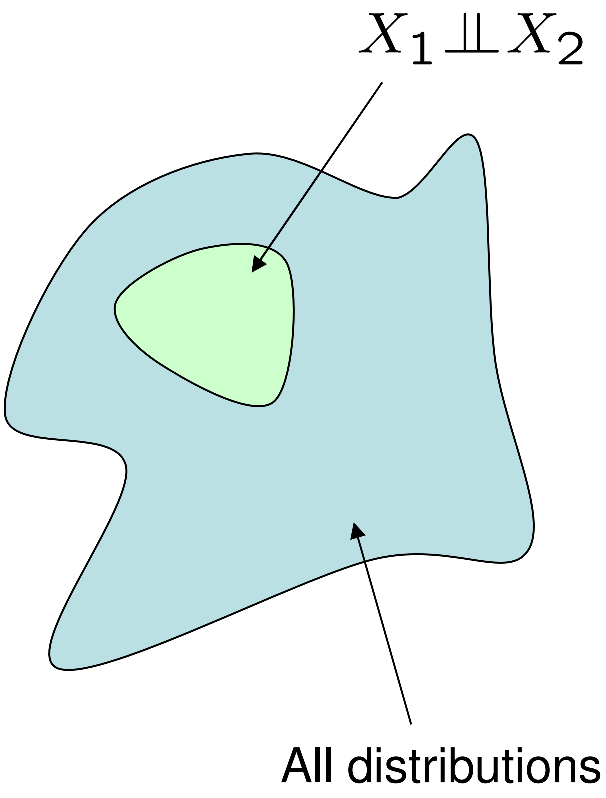
- For this graph, you can fiddle with θ (the CPTs) all you want, but you won't be able to represent any distribution in which the flips are dependent!

$P(X_1)$

h	0.5
t	0.5

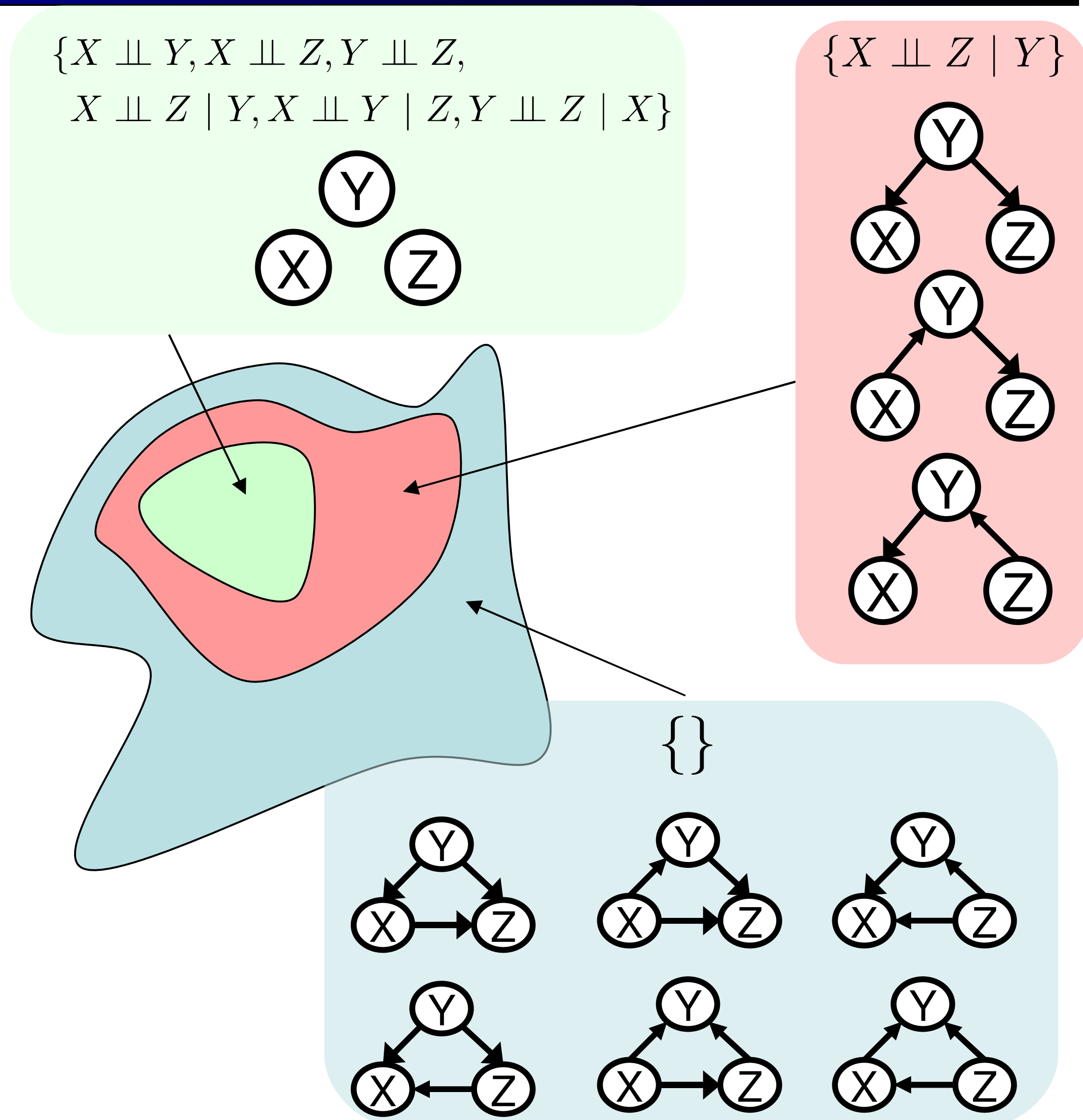
$P(X_2)$

h	0.5
t	0.5



Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts

- BNs need not actually be causal
  - Sometimes no causal net exists over the domain
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation

- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology only guaranteed to encode conditional independence

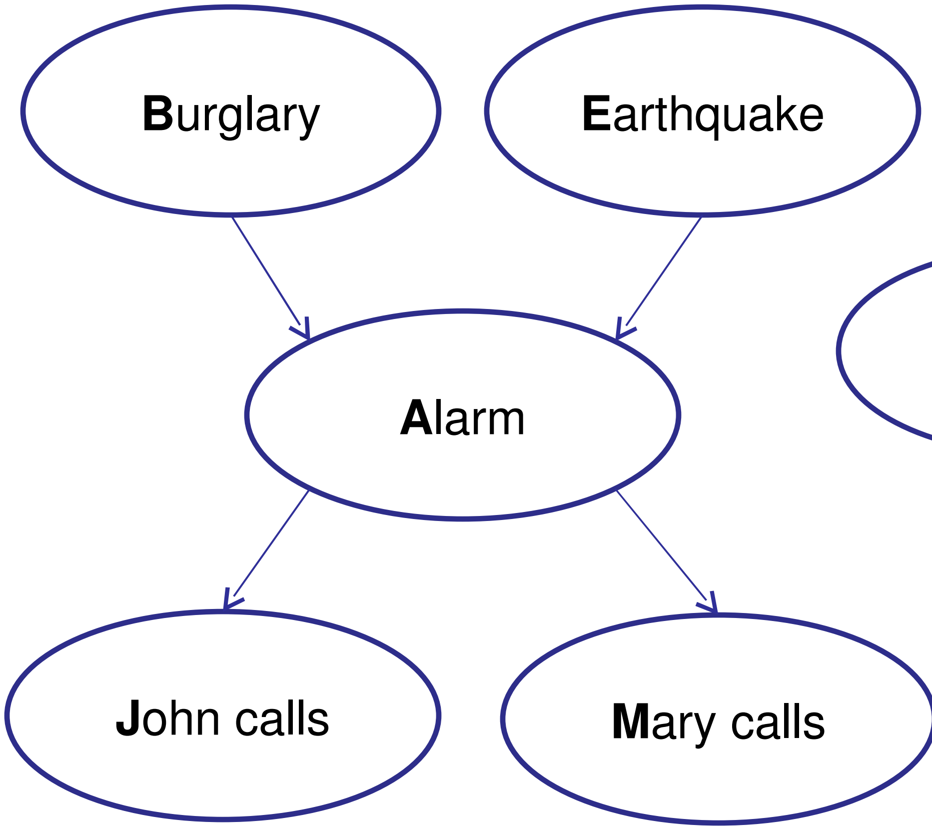
- *More about causality: [Causility – Judea Pearl]

Changing Bayes’ Net Structure

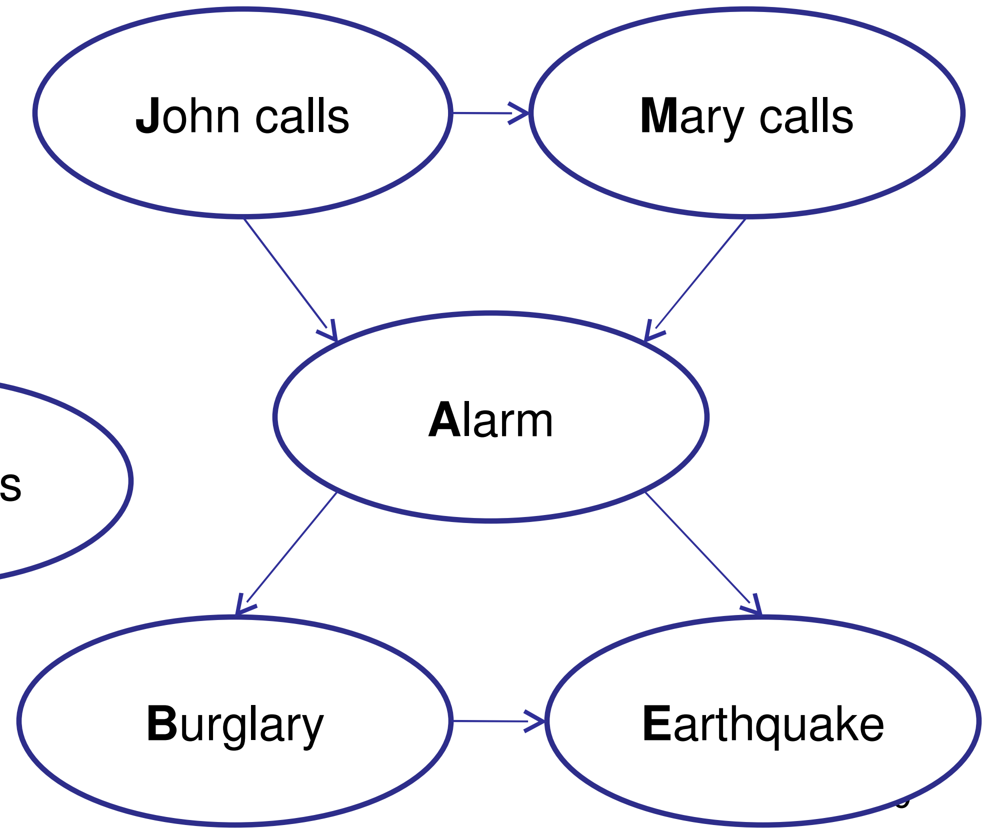
- The same joint distribution can be encoded in many different Bayes’ nets
  - Causal structure tends to be the simplest

- Analysis question: given some edges, what other edges do you need to add?
  - One answer: fully connect the graph
  - Better answer: don’t make any false conditional independence assumptions

Example: Alternate Alarm



If we reverse the edges, we make different conditional independence assumptions



To capture the same joint distribution, we have to add more edges to the graph

Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- Can analyze precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

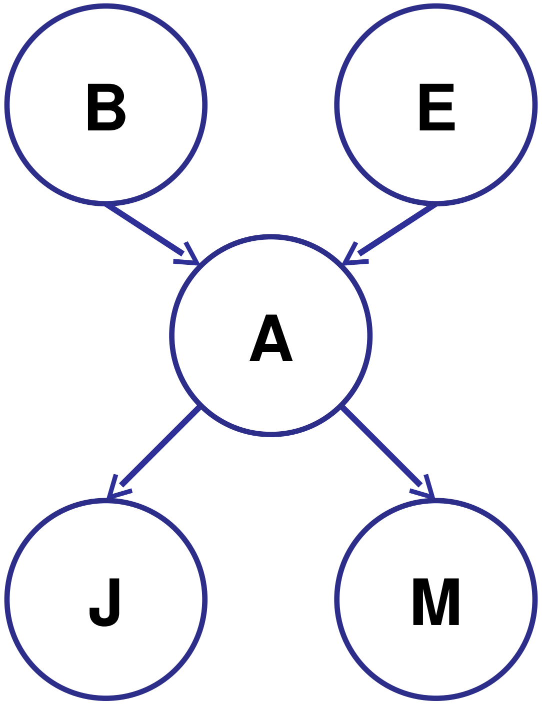
Inference

- Inference: calculating some useful quantity from a joint probability distribution
- Examples:
  - Posterior probability:

$$P(Q|E_1 = e_1, \ldots E_k = e_k)$$

  - Most likely explanation:

$$\text{argmax}_q \, P(Q = q|E_1 = e_1 \ldots)$$

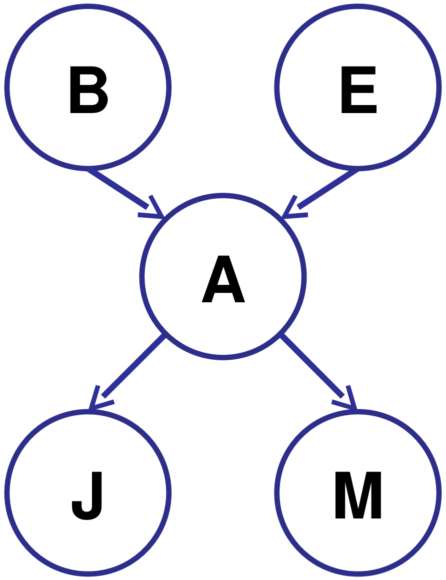


Inference by Enumeration

- Given unlimited time, inference in BNs is easy
- Recipe:
  - State the marginal probabilities you need
  - Figure out ALL the atomic probabilities you need
  - Calculate and combine them
- Example:

$$P(+b|+j,+m) =$$

$$\frac{P(+b,+j,+m)}{P(+j,+m)}$$



Example: Enumeration

- In this simple method, we only need the BN to synthesize the joint entries

$$P(+b,+j,+m) =$$

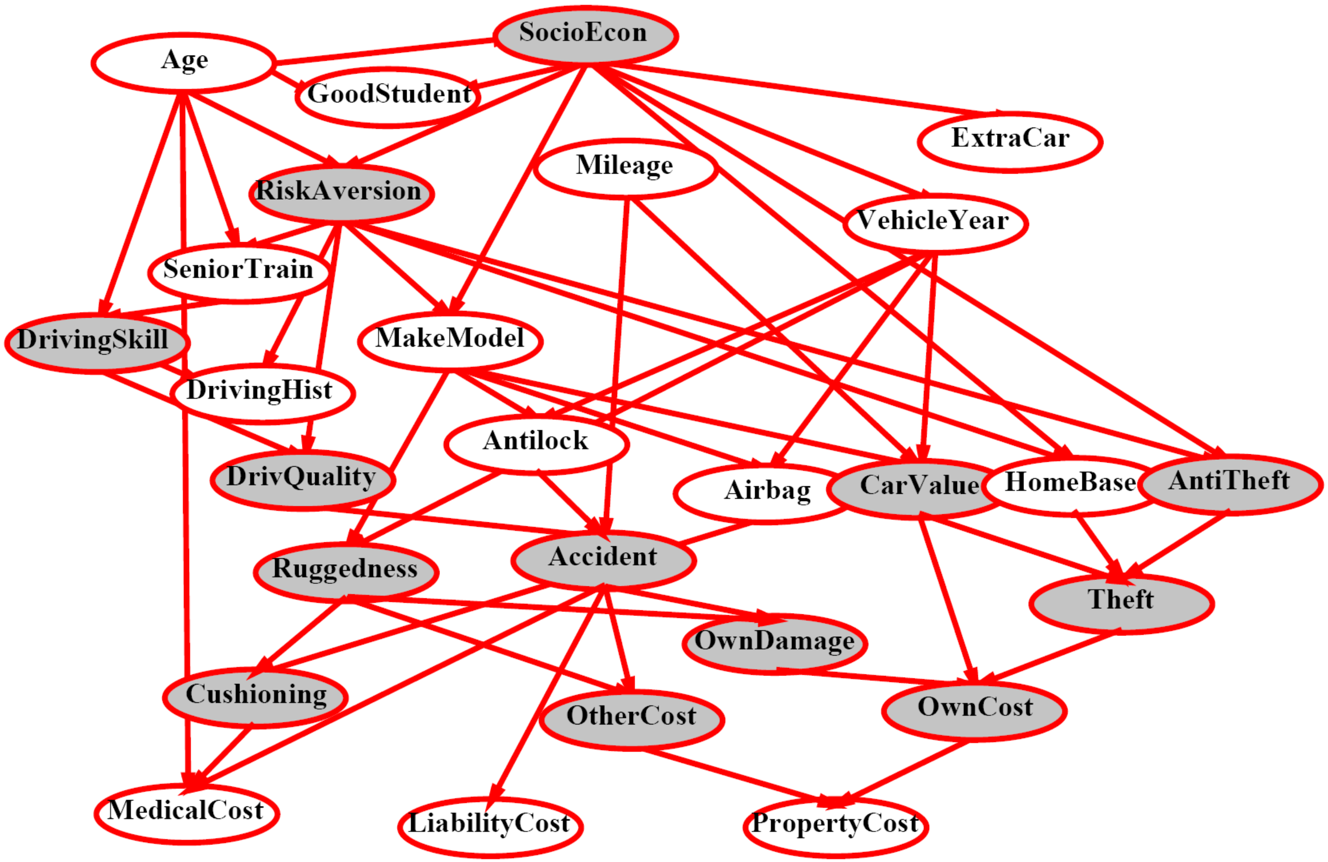
$$P(+b)P(+e)P(+a|+b,+e)P(+j|+a)P(+m|+a)+$$

$$P(+b)P(+e)P(-a|+b,+e)P(+j|-a)P(+m|-a)+$$

$$P(+b)P(-e)P(+a|+b,-e)P(+j|+a)P(+m|+a)+$$

$$P(+b)P(-e)P(-a|+b,-e)P(+j|-a)P(+m|-a)$$

Inference by Enumeration?



Variable Elimination

- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables
  - You end up repeating a lot of work!
- Idea: interleave joining and marginalizing!
  - Called “Variable Elimination”
  - Still NP-hard, but usually much faster than inference by enumeration
- We’ll need some new notation to define VE

Factor Zoo I

- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - Sums to 1

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Selected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for fixed x, all y
  - Sums to P(x)

$P(cold, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

Factor Zoo II

- Family of conditionals: P(X |Y)
  - Multiple conditionals
  - Entries P(x | y) for all x, y
  - Sums to |Y|

$P(W|T)$

T	W	P
hot	sun	0.8
hot	rain	0.2
cold	sun	0.4
cold	rain	0.6

$P(W|hot)$

$P(W|cold)$

- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, all y
  - Sums to 1

$P(W|cold)$

T	W	P
cold	sun	0.4
cold	rain	0.6

Factor Zoo III

- Specified family: P(y | X)
  - Entries P(y | x) for fixed y, but for all x
  - Sums to … who knows!

$P(rain|T)$

T	W	P
hot	rain	0.2
cold	rain	0.6

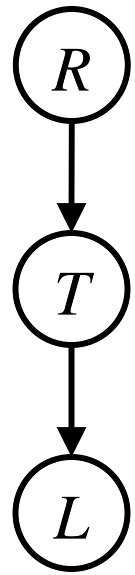
$\} P(rain|hot)$

$\} P(rain|cold)$

- In general, when we write $P(Y_1 \dots Y_N \mid X_1 \dots X_M)$
  - It is a "factor," a multi-dimensional array
  - Its values are all $P(y_1 \dots y_N \mid x_1 \dots x_M)$
  - Any assigned X or Y is a dimension missing (selected) from the array

Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!
- First query: P(L)



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|R)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Variable Elimination Outline

- Track objects called factors
- Initial factors are local CPTs (one per node)

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

- Any known values are selected
  - E.g. if we know $L = +\ell$, the initial factors are

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(+\ell|T)$

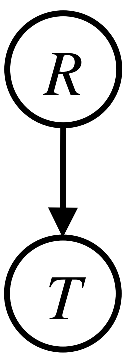
+t	+l	0.3
-t	+l	0.1

- VE: Alternately join factors and eliminate variables

Operation 1: Join Factors

- First basic operation: joining factors
- Combining factors:
  - Just like a database join
  - Get all factors over the joining variable
  - Build a new factor over the union of the variables involved

- Example: Join on R

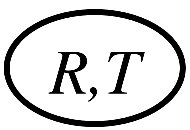


$$P(R) \quad \times \quad P(T|R) \quad \Rightarrow \quad P(R,T)$$

+r	0.1
-r	0.9

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

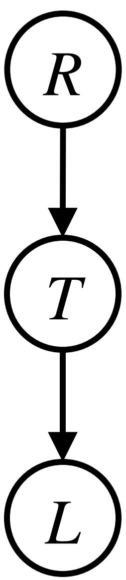


- Computation for each entry: pointwise products

$$\forall r,t: \quad P(r,t) = P(r) \cdot P(t|r)$$

Operation 1: Join Factors

- In general, we join on a variable
  - Take all factors mentioning that variable
  - Join them all together with pointwise products
  - Result is P(all LHS vars | all non-LHS vars)
  - Leave other factors alone
- Example: Join on T



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

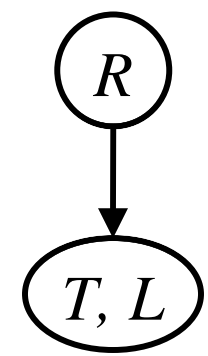
+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

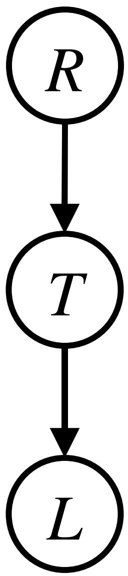
+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

$P(T, L|R)$

+r	+t	+l	0.24
+r	+t	-l	0.56
+r	-t	+l	0.02
+r	-t	-l	0.18
-r	+t	+l	0.03
-r	+t	-l	0.07
-r	-t	+l	0.09
-r	-t	-l	0.81



Example: Multiple Joins



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

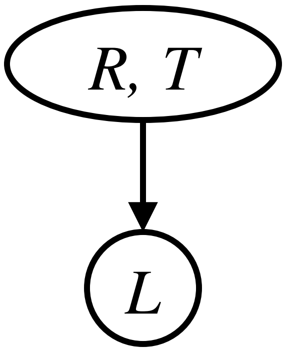
Join R

$P(R,T)$

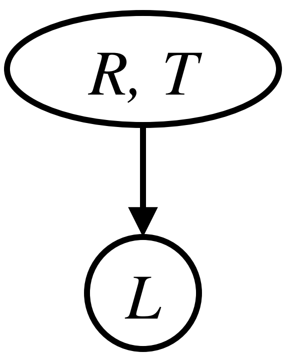
+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9



Example: Multiple Joins



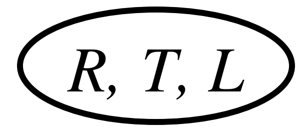
$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join T



$P(R,T,L)$

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

Operation 2: Eliminate

- Second basic operation: marginalization
- Take a factor and sum out a variable
  - Shrinks a factor to a smaller one
  - A projection operation
- Example:

$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

sum R $\longrightarrow$

$P(T)$

+t	0.17
-t	0.83

Multiple Elimination

R, T, L | T, L | L

$P(R, T, L)$

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

Sum out R

$P(T, L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Sum out T

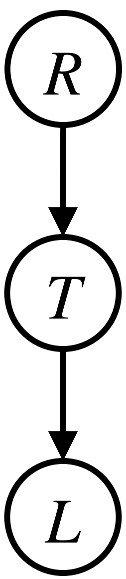
$P(L)$

+l	0.134
-l	0.886

P(L) : Marginalizing Early!

$P(R)$

+r	0.1
-r	0.9



$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

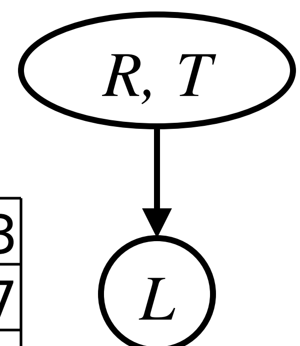
Join R

$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

$P(L|T)$

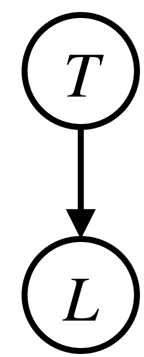
+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9



Sum out R

$P(T)$

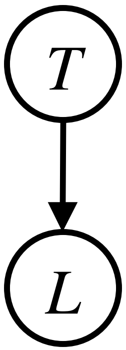
+t	0.17
-t	0.83



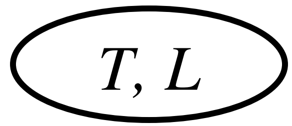
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

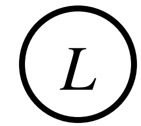
Marginalizing Early (aka VE*)



Join T



Sum out T



$P(T)$

+t	0.17
-t	0.83

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

$P(T, L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

$P(L)$

+l	0.134
-l	0.886

* VE is variable elimination

Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

  - Computing $P(L|+r)$, the initial factors become:

$P(+r)$

+r	0.1

$P(T|+r)$

+r	+t	0.8
+r	-t	0.2

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

- We eliminate all vars other than query + evidence

Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we'd end up with:

$P(+r, L)$

+r	+l	0.026
+r	-l	0.074

Normalize

$P(L| + r)$

+l	0.26
-l	0.74

- To get our answer, just normalize this!

- That's it!

General Variable Elimination

- Query: $P(Q|E_1 = e_1, \ldots E_k = e_k)$

- Start with initial factors:
  - Local CPTs (but instantiated by evidence)

- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H

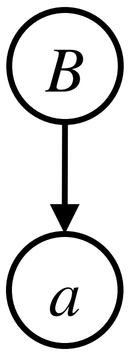
- Join all remaining factors and normalize

Variable Elimination Bayes Rule

Start / Select

$P(B)$

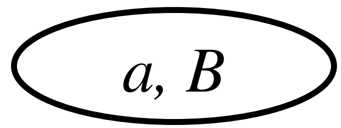
B	P
+b	0.1
¬b	0.9



$P(A|B) \rightarrow P(a|B)$

B	A	P
+b	+a	0.8
~~+b~~	~~¬a~~	~~0.2~~
¬b	+a	0.1
~~¬b~~	~~¬a~~	~~0.9~~

Join on B



$P(a, B)$

A	B	P
+a	+b	0.08
+a	¬b	0.09

Normalize

$P(B|a)$

A	B	P
+a	+b	8/17
+a	¬b	9/17

Example

$$P(B|j,m) \propto P(B,j,m)$$

$$\boxed{P(B) \qquad P(E) \qquad P(A|B,E) \qquad P(j|A) \qquad P(m|A)}$$

Choose A

$$\left.\begin{array}{l} P(A|B,E) \\ P(j|A) \\ P(m|A) \end{array}\right. \xrightarrow{\times} P(j,m,A|B,E) \xrightarrow{\sum} P(j,m|B,E)$$

$$\boxed{P(B) \qquad P(E) \qquad P(j,m|B,E)}$$

Example

$$P(B) \qquad P(E) \qquad P(j,m|B,E)$$

Choose E

$$P(E),\ P(j,m|B,E) \xrightarrow{\times} P(j,m,E|B) \xrightarrow{\sum} P(j,m|B)$$

$$P(B) \qquad P(j,m|B)$$

Finish with B

$$P(B),\ P(j,m|B) \xrightarrow{\times} P(j,m,B) \xrightarrow{\text{Normalize}} P(B|j,m)$$

Variable Elimination

- What you need to know:
  - Should be able to run it on small examples, understand the factor creation / reduction flow
  - Better than enumeration: saves time by marginalizing variables as soon as possible rather than at the end
- We will see special cases of VE later
  - On tree-structured graphs, variable elimination runs in polynomial time
  - You'll have to implement a tree-structured special case to track invisible ghosts (Project 4)