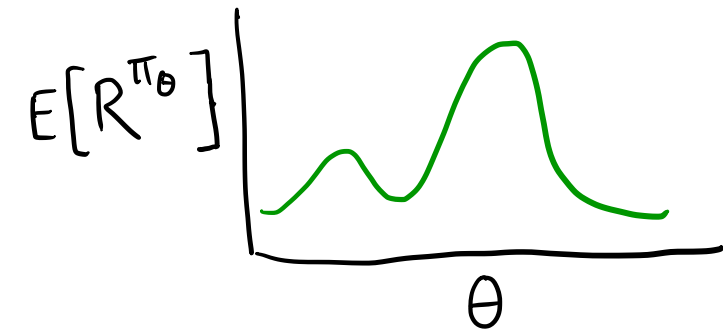


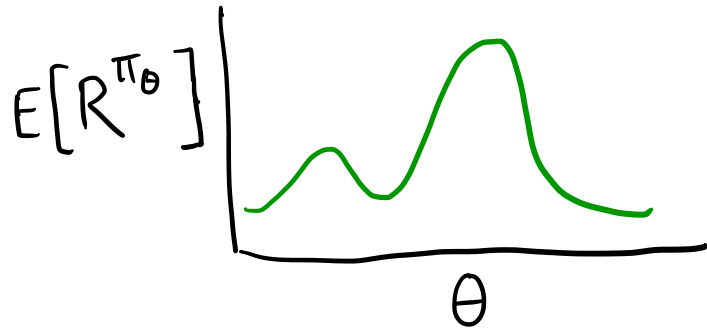
Policy Search

Hill Climbing



Policy Search

Hill Climbing



Genetic Search

$$\theta_1 \quad \boxed{\theta_1^1 \mid \theta_1^2 \mid \theta_1^3 \mid \theta_1^4}$$

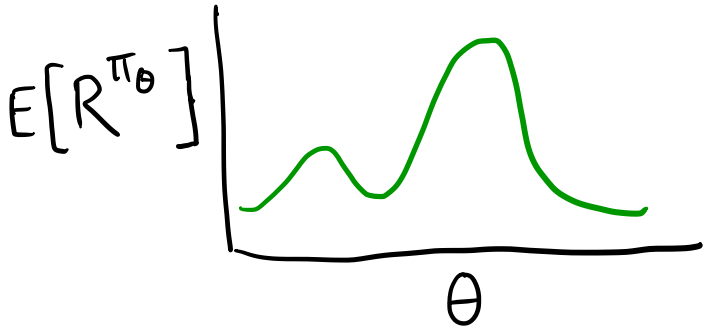
$$\theta_2 \quad \boxed{\theta_2^1 \mid \theta_2^2 \mid \theta_2^3 \mid \theta_2^4}$$

:

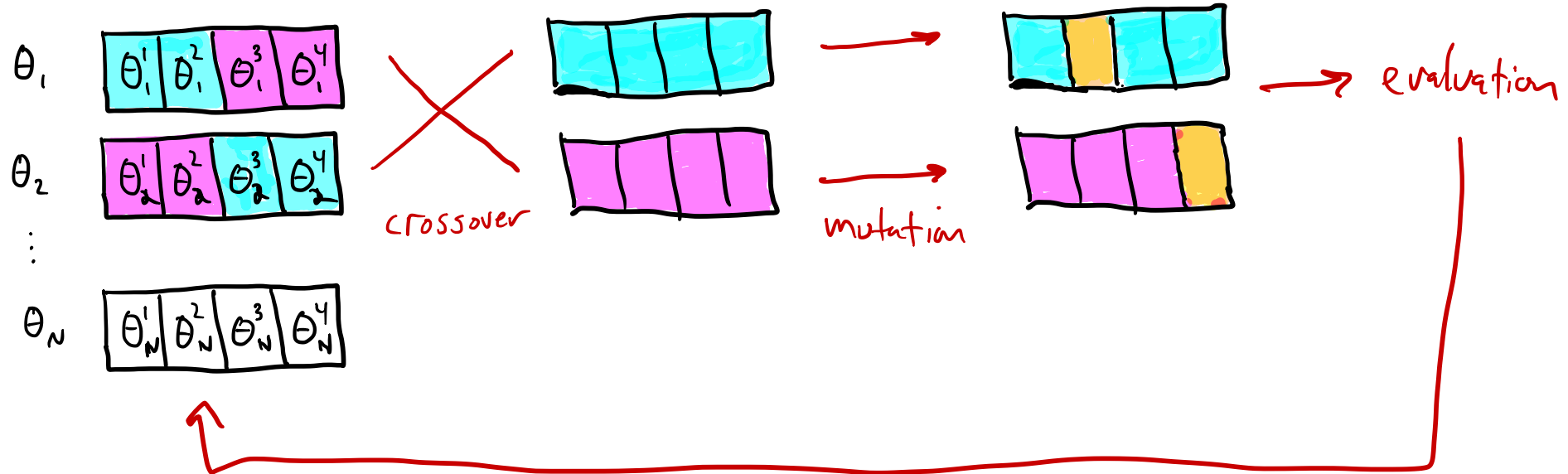
$$\theta_N \quad \boxed{\theta_N^1 \mid \theta_N^2 \mid \theta_N^3 \mid \theta_N^4}$$

Policy Search

Hill Climbing



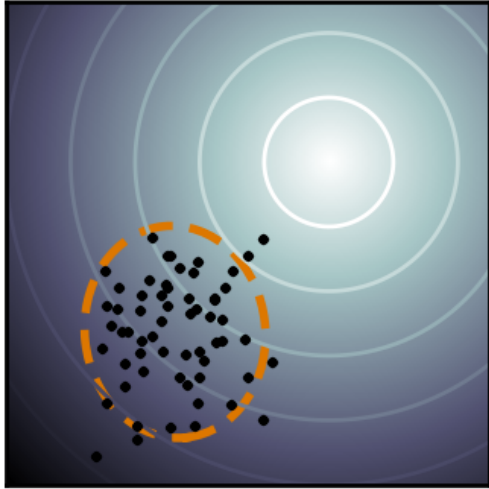
Genetic Search



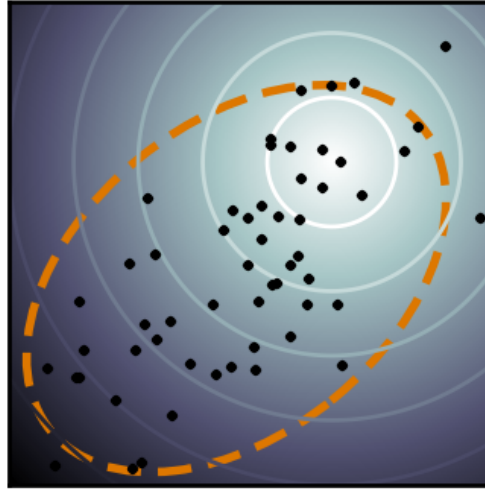
Policy Search

CMA-ES

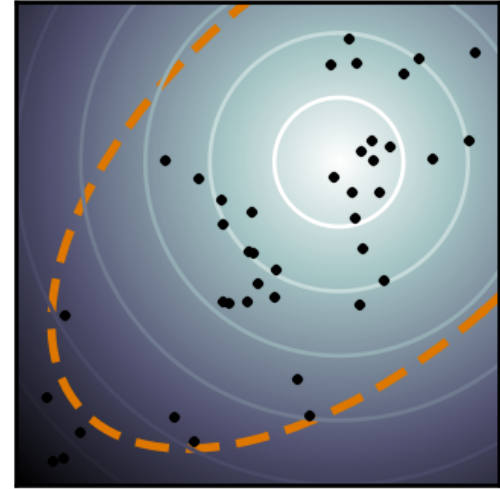
Generation 1



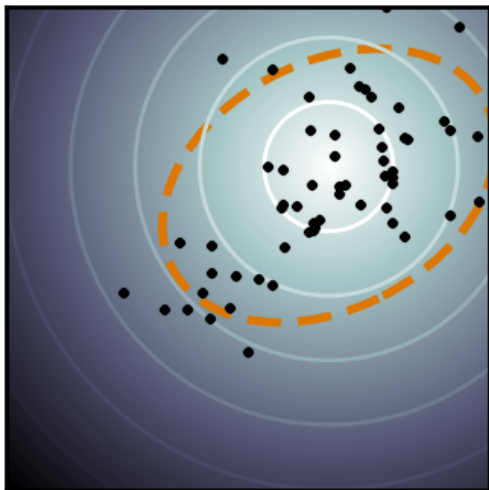
Generation 2



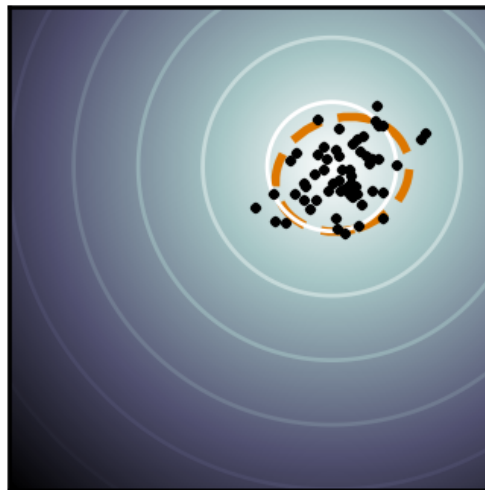
Generation 3



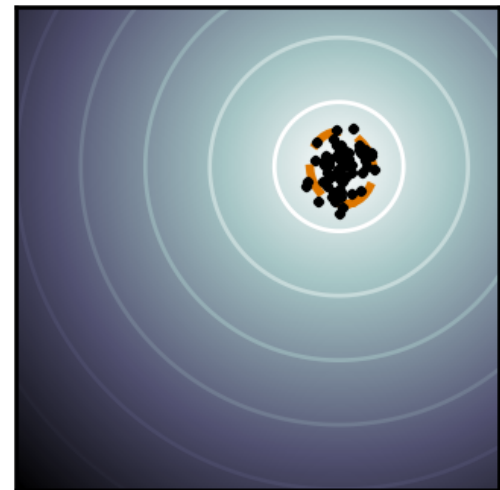
Generation 4



Generation 5



Generation 6



Gradient Bandits :

$$H_{t+1}(a) \doteq H_t(a) + \alpha \frac{\partial \mathbb{E}[R_t]}{\partial H_t(a)}$$

$$\frac{\partial \mathbb{E}[R_t]}{\partial H_t(a)} = \frac{\partial}{\partial H_t(a)} \left[\sum_x \pi_t(x) q_*(x) \right]$$

Gradient Bandits:

$$H_{t+1}(a) \doteq H_t(a) + \alpha \frac{\partial \mathbb{E}[R_t]}{\partial H_t(a)}$$

$$\frac{\partial \mathbb{E}[R_t]}{\partial H_t(a)} = \frac{\partial}{\partial H_t(a)} \left[\sum_x \pi_t(x) q_*(x) \right]$$

Just a scalar per arm.
No states!

But in full RL case, policy influences future states!

Proof of policy gradient theorem

$$\begin{aligned}
\nabla v_\pi(s) &= \nabla \left[\sum_a \pi(a|s) q_\pi(s, a) \right], \quad \text{for all } s \in \mathcal{S} && \text{(Exercise 3.18)} \\
&= \sum_a \left[\nabla \pi(a|s) q_\pi(s, a) + \pi(a|s) \nabla q_\pi(s, a) \right] && \text{(product rule of calculus)} \\
&= \sum_a \left[\nabla \pi(a|s) q_\pi(s, a) + \pi(a|s) \nabla \sum_{s', r} p(s', r | s, a) (r + v_\pi(s')) \right] \\
&&& \text{(Exercise 3.19 and Equation 3.2)} \\
&= \sum_a \left[\nabla \pi(a|s) q_\pi(s, a) + \pi(a|s) \sum_{s'} p(s' | s, a) \nabla v_\pi(s') \right] && \text{(Eq. 3.4)} \\
&= \sum_a \left[\nabla \pi(a|s) q_\pi(s, a) + \pi(a|s) \sum_{s'} p(s' | s, a) \right. \\
&\quad \left. \sum_{a'} [\nabla \pi(a' | s') q_\pi(s', a') + \pi(a' | s') \sum_{s''} p(s'' | s', a') \nabla v_\pi(s'')] \right] \\
&= \sum_{x \in \mathcal{S}} \sum_{k=0}^{\infty} \Pr(s \rightarrow x, k, \pi) \sum_a \nabla \pi(a|x) q_\pi(x, a),
\end{aligned}$$

after repeated unrolling, where $\Pr(s \rightarrow x, k, \pi)$ is the probability of transitioning from state s to state x in k steps under policy π . It is then immediate that

$$\begin{aligned}
\nabla J(\boldsymbol{\theta}) &= \nabla v_\pi(s_0) \\
&= \sum_s \left(\sum_{k=0}^{\infty} \Pr(s_0 \rightarrow s, k, \pi) \right) \sum_a \nabla \pi(a|s) q_\pi(s, a) \\
&= \sum_s \eta(s) \sum_a \nabla \pi(a|s) q_\pi(s, a) && \text{(box page 199)} \\
&= \sum_{s'} \eta(s') \sum_s \frac{\eta(s)}{\sum_{s'} \eta(s')} \sum_a \nabla \pi(a|s) q_\pi(s, a) \\
&= \sum_{s'} \eta(s') \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s, a) && \text{(Eq. 9.3)} \\
&\propto \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s, a) && \text{(Q.E.D.)}
\end{aligned}$$

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\end{aligned}$$

Proof of policy gradient theorem

Marginalize R, push in gradient
Dynamics + Reward constant
w.r.t. θ

after repeated unrolling, where $\text{Pr}(s \rightarrow x, k, \pi)$ is the probability of transitioning from state s to state x in k steps under policy π . It is then immediate that

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&= \sum_s \eta(s) \sum_a \nabla \pi(a|s) q_\pi(s, a) && \text{(box page 199)} \\
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Marginalize R, push in gradient
Dynamics + Reward constant
w.r.t. θ

Expanding $v_\pi(s')$ creates
deeply nested computation:

after repeated unrolling, where $\text{Pr}(s \rightarrow x, k, \pi)$ is the probability of transitioning from state s to state x in k steps under policy π . It is then immediate that

At every step, compute every
state you could get to from
every state you could have been in

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 &= \sum_{s'} \eta(s') \sum_s \frac{\eta(s)}{\sum_{s'} \eta(s')} \sum_a \nabla \pi(a|s) q_\pi(s, a) \\
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 \end{aligned}$$

↓
Transform into simple sum over
time steps and states:

What is total prob of being at
each state at each time step?

Proof of policy gradient theorem

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 & \quad \left. \sum_{a'} [\nabla \pi(a' | s') q_\pi(s', a') + \pi(a' | s') \sum_{s''} p(s'' | s', a') \nabla v_\pi(s'')] \right] && \text{(unrolling)} \\
 &= \sum_{x \in \mathcal{S}} \sum_{k=0}^{\infty} \text{Pr}(s \rightarrow x, k, \pi) \sum_a \nabla \pi(a|x) q_\pi(x, a),
 \end{aligned}$$

Marginalize R, push in gradient
Dynamics + Reward constant
w.r.t. θ

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 \end{aligned}$$

unnormalized
steady-state
prob of s

normalized
version

↓
Transform into simple sum over
time steps and states:

What is total prob of being at
each state at each time step?

REINFORCE

$$\begin{aligned}\nabla J(\boldsymbol{\theta}) &\propto \sum_s \mu(s) \sum_a q_\pi(s, a) \nabla \pi(a|s, \boldsymbol{\theta}) \\ &= \mathbb{E}_\pi \left[\sum_a q_\pi(S_t, a) \nabla \pi(a|S_t, \boldsymbol{\theta}) \right].\end{aligned}$$



$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha \sum_a \hat{q}(S_t, a, \mathbf{w}) \nabla \pi(a|S_t, \boldsymbol{\theta})$$

All actions \rightarrow \uparrow \leftarrow Q approx, not a sample return

REINFORCE

$$\begin{aligned}\nabla J(\boldsymbol{\theta}) &\propto \sum_s \mu(s) \sum_a q_\pi(s, a) \nabla \pi(a|s, \boldsymbol{\theta}) \\ &= \mathbb{E}_\pi \left[\sum_a q_\pi(S_t, a) \nabla \pi(a|S_t, \boldsymbol{\theta}) \right].\end{aligned}$$



$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha \sum_a \hat{q}(S_t, a, \mathbf{w}) \nabla \pi(a|S_t, \boldsymbol{\theta})$$

$$\nabla J(\boldsymbol{\theta}) = \mathbb{E}_\pi \left[\sum_a \pi(a|S_t, \boldsymbol{\theta}) q_\pi(S_t, a) \frac{\nabla \pi(a|S_t, \boldsymbol{\theta})}{\pi(a|S_t, \boldsymbol{\theta})} \right]$$

$$= \mathbb{E}_\pi \left[q_\pi(S_t, A_t) \frac{\nabla \pi(A_t|S_t, \boldsymbol{\theta})}{\pi(A_t|S_t, \boldsymbol{\theta})} \right]$$

(replacing a by the sample $A_t \sim \pi$)

$$= \mathbb{E}_\pi \left[G_t \frac{\nabla \pi(A_t|S_t, \boldsymbol{\theta})}{\pi(A_t|S_t, \boldsymbol{\theta})} \right],$$

(because $\mathbb{E}_\pi[G_t|S_t, A_t] = q_\pi(S_t, A_t)$)

REINFORCE

$$\begin{aligned}\nabla J(\boldsymbol{\theta}) &\propto \sum_s \mu(s) \sum_a q_\pi(s, a) \nabla \pi(a|s, \boldsymbol{\theta}) \\ &= \mathbb{E}_\pi \left[\sum_a q_\pi(S_t, a) \nabla \pi(a|S_t, \boldsymbol{\theta}) \right].\end{aligned} \quad \longrightarrow \quad \boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha \sum_a \hat{q}(S_t, a, \mathbf{w}) \nabla \pi(a|S_t, \boldsymbol{\theta})$$

$$\begin{aligned}\nabla J(\boldsymbol{\theta}) &= \mathbb{E}_\pi \left[\sum_a \pi(a|S_t, \boldsymbol{\theta}) q_\pi(S_t, a) \frac{\nabla \pi(a|S_t, \boldsymbol{\theta})}{\pi(a|S_t, \boldsymbol{\theta})} \right] \\ &= \mathbb{E}_\pi \left[q_\pi(S_t, A_t) \frac{\nabla \pi(A_t|S_t, \boldsymbol{\theta})}{\pi(A_t|S_t, \boldsymbol{\theta})} \right] && \text{(replacing } a \text{ by the sample } A_t \sim \pi) \\ &= \mathbb{E}_\pi \left[G_t \frac{\nabla \pi(A_t|S_t, \boldsymbol{\theta})}{\pi(A_t|S_t, \boldsymbol{\theta})} \right], && \text{(because } \mathbb{E}_\pi[G_t|S_t, A_t] = q_\pi(S_t, A_t))\end{aligned}$$

REINFORCE: Monte-Carlo Policy-Gradient Control (episodic) for π_*

Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$

Algorithm parameter: step size $\alpha > 0$

Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):

Generate an episode $S_0, A_0, R_1, \dots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot, \boldsymbol{\theta})$

Loop for each step of the episode $t = 0, 1, \dots, T - 1$:

$$\begin{aligned}G &\leftarrow \sum_{k=t+1}^T \gamma^{k-t-1} R_k && (G_t) \\ \boldsymbol{\theta} &\leftarrow \boldsymbol{\theta} + \alpha \gamma^t G \nabla \ln \pi(A_t|S_t, \boldsymbol{\theta})\end{aligned}$$

Gradient Bandits + Baseline

$$H_{t+1}(a) \doteq H_t(a) + \alpha \frac{\partial \mathbb{E}[R_t]}{\partial H_t(a)}$$

$$\frac{\partial \mathbb{E}[R_t]}{\partial H_t(a)} = \frac{\partial}{\partial H_t(a)} \left[\sum_x \pi_t(x) q_*(x) \right]$$

$$= \sum_x q_*(x) \frac{\partial \pi_t(x)}{\partial H_t(a)}$$

$$= \sum_x (q_*(x) - B_t) \frac{\partial \pi_t(x)}{\partial H_t(a)},$$

Mean
of
Samples

Expectation
zero

Gradient Bandits + Baseline

$$H_{t+1}(a) \doteq H_t(a) + \alpha \frac{\partial \mathbb{E}[R_t]}{\partial H_t(a)}$$

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$$= \sum_x (q_*(x) - B_t) \frac{\partial \pi_t(x)}{\partial H_t(a)},$$

Mean
of
Samples

Expectation
zero

REINFORCE + Baseline

$$\nabla J(\theta) \propto \sum_s \mu(s) \sum_a q_\pi(s, a) \nabla \pi(a|s, \theta)$$



$$\nabla J(\theta) \propto \sum_s \mu(s) \sum_a (q_\pi(s, a) - b(s)) \nabla \pi(a|s, \theta).$$

$$\sum_a b(s) \nabla \pi(a|s, \theta) = b(s) \nabla \sum_a \pi(a|s, \theta) = b(s) \nabla 1 = 0.$$

$$\theta_{t+1} \doteq \theta_t + \alpha \left(G_t - \boxed{b(S_t)} \right) \frac{\nabla \pi(A_t|S_t, \theta_t)}{\pi(A_t|S_t, \theta_t)}.$$



$\hat{V}(S_t)$

REINFORCE with Baseline (episodic), for estimating $\pi_{\theta} \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s, \theta)$

Input: a differentiable state-value function parameterization $\hat{v}(s, \mathbf{w})$

Algorithm parameters: step sizes $\alpha^{\theta} > 0$, $\alpha^{\mathbf{w}} > 0$

Initialize policy parameter $\theta \in \mathbb{R}^d$ and state-value weights $\mathbf{w} \in \mathbb{R}^d$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):

 Generate an episode $S_0, A_0, R_1, \dots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot, \theta)$

 Loop for each step of the episode $t = 0, 1, \dots, T - 1$:

$$G \leftarrow \sum_{k=t+1}^T \gamma^{k-t-1} R_k \quad (G_t)$$

$$\delta \leftarrow G - \hat{v}(S_t, \mathbf{w})$$

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} \delta \nabla \hat{v}(S_t, \mathbf{w})$$

$$\theta \leftarrow \theta + \alpha^{\theta} \gamma^t \delta \nabla \ln \pi(A_t|S_t, \theta)$$

One-step Actor-Critic (episodic), for estimating $\pi_{\theta} \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s, \theta)$

Input: a differentiable state-value function parameterization $\hat{v}(s, \mathbf{w})$

Parameters: step sizes $\alpha^{\theta} > 0$, $\alpha^{\mathbf{w}} > 0$

Initialize policy parameter $\theta \in \mathbb{R}^d$ and state-value weights $\mathbf{w} \in \mathbb{R}^d$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):

 Initialize S (first state of episode)

$I \leftarrow 1$

 Loop while S is not terminal (for each time step):

$A \sim \pi(\cdot|S, \theta)$

 Take action A , observe S', R

$\delta \leftarrow R + \gamma \hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w})$ (if S' is terminal, then $\hat{v}(S', \mathbf{w}) \doteq 0$)

$\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} \delta \nabla \hat{v}(S, \mathbf{w})$

$\theta \leftarrow \theta + \alpha^{\theta} I \delta \nabla \ln \pi(A|S, \theta)$

$I \leftarrow \gamma I$

$S \leftarrow S'$

Actor only

- policy search
- Directly parameterized policy
- No value functions (except baseline in REINFORCE)
- Continuous actions natural to represent
- High variance, No bootstrapping
- Scales w/ policy complexity, not size of state space

Actor only

- policy search
- Directly parameterized policy
- No value functions (except baseline in REINFORCE)
- Continuous actions natural to represent
- High variance, No bootstrapping
- Scales w/ policy complexity, not size of state space

Critic only

- value function methods
- Indirect policy via VF
- Discrete actions only
- Lower variance, bootstrapping
- Scales with size of state space

Actor only

- policy search
- Directly parameterized policy
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Critic only

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Actor - Critic

- Policy Search + value function
- Benefits of both!
- Continuous actions
- Bootstrapping
- Scales primarily with policy complexity

Actor only

- policy search
- Directly parameterized policy
- No value functions (except baseline in REINFORCE)
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Many of most popular contemporary methods are A-C:

- Proximal Policy Optimization
- A3C
- Soft Actor Critic
- DDPG

⋮