Lecture 5:
Basic Dynamical Systems

CS 344R: Robotics

Benjamin Kuipers

Dynamical Systems

- A *dynamical system* changes continuously (almost always) according to

$$\dot{\mathbf{x}} = F(\mathbf{x}) \quad \text{where} \quad \mathbf{x} \in \Re^n$$

- A *controller* is defined to change the coupled robot and environment into a desired dynamical system.

$$\dot{\mathbf{x}} = F(\mathbf{x}, \mathbf{u})$$

$$\mathbf{y} = G(\mathbf{x})$$

$$\mathbf{u} = H_i(\mathbf{y})$$

$$\dot{\mathbf{x}} = F(\mathbf{x}, H_i(G(\mathbf{x})))$$

In One Dimension

- Simple linear system

  $$\dot{x} = kx$$

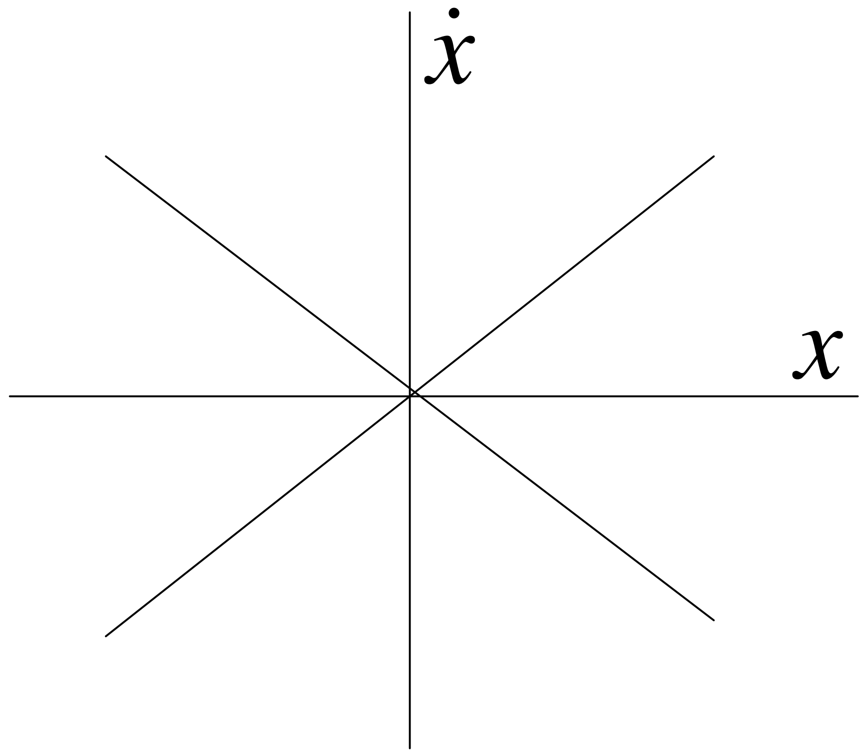
- Fixed point

  $$x = 0 \quad \Rightarrow \quad \dot{x} = 0$$

- Solution

  $$x(t) = x_0 \, e^{kt}$$

  - Stable if $k < 0$
  - Unstable if $k > 0$



In Two Dimensions

- Often, position and velocity:

$$\mathbf{x} = (x, v)^T \quad \text{where} \quad v = \dot{x}$$

- If actions are forces, causing acceleration:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} v \\ forces \end{pmatrix}$$

The Damped Spring

- The spring is defined by Hooke's Law:

$$F = ma = m\ddot{x} = -k_1 x$$

- Include damping friction

$$m\ddot{x} = -k_1 x - k_2 \dot{x}$$

- Rearrange and redefine constants

$$\ddot{x} + b\dot{x} + cx = 0$$

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} v \\ -b\dot{x} - cx \end{pmatrix}$$

The Linear Spring Model

$$\ddot{x} + b\dot{x} + cx = 0 \qquad\qquad c \neq 0$$

- Solutions are:

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

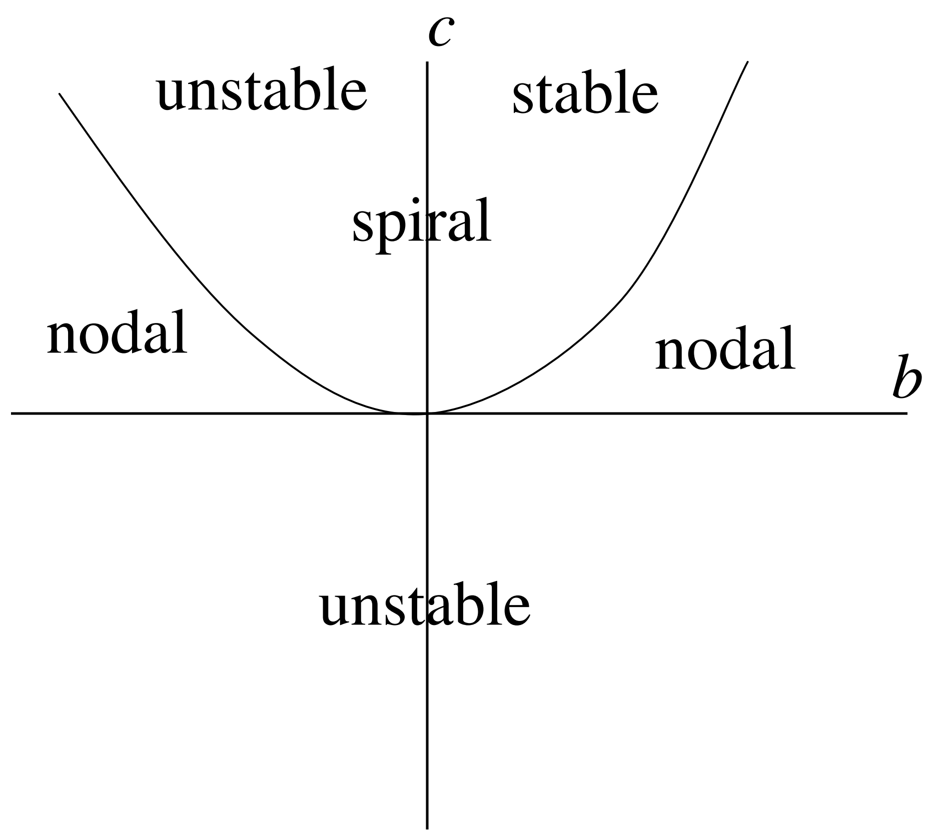
- Where r_1, r_2 are roots of the characteristic equation $\lambda^2 + b\lambda + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Qualitative Behaviors

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

- $\mathrm{Re}(r_1), \mathrm{Re}(r_2) < 0$ means stable.
- $\mathrm{Re}(r_1), \mathrm{Re}(r_2) > 0$ means unstable.
- b^2-4$c < 0$ means complex roots, means oscillations.



Generalize to Higher Dimensions

- The characteristic equation for $\dot{\mathbf{x}} = A\mathbf{x}$ generalizes to $\det(A - \lambda I) = 0$
  - This means that there is a vector **v** such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

- The solutions λ are called *eigenvalues*.
- The related vectors **v** are *eigenvectors*.

Qualitative Behavior, Again

- For a dynamical system to be stable:
  - The real parts of all eigenvalues must be negative.
  - All eigenvalues lie in the left half complex plane.
- Terminology:
  - *Underdamped* = spiral (some complex eigenvalue)
  - *Overdamped* = nodal (all eigenvalues real)
  - *Critically damped* = the boundary between.

Node Behavior

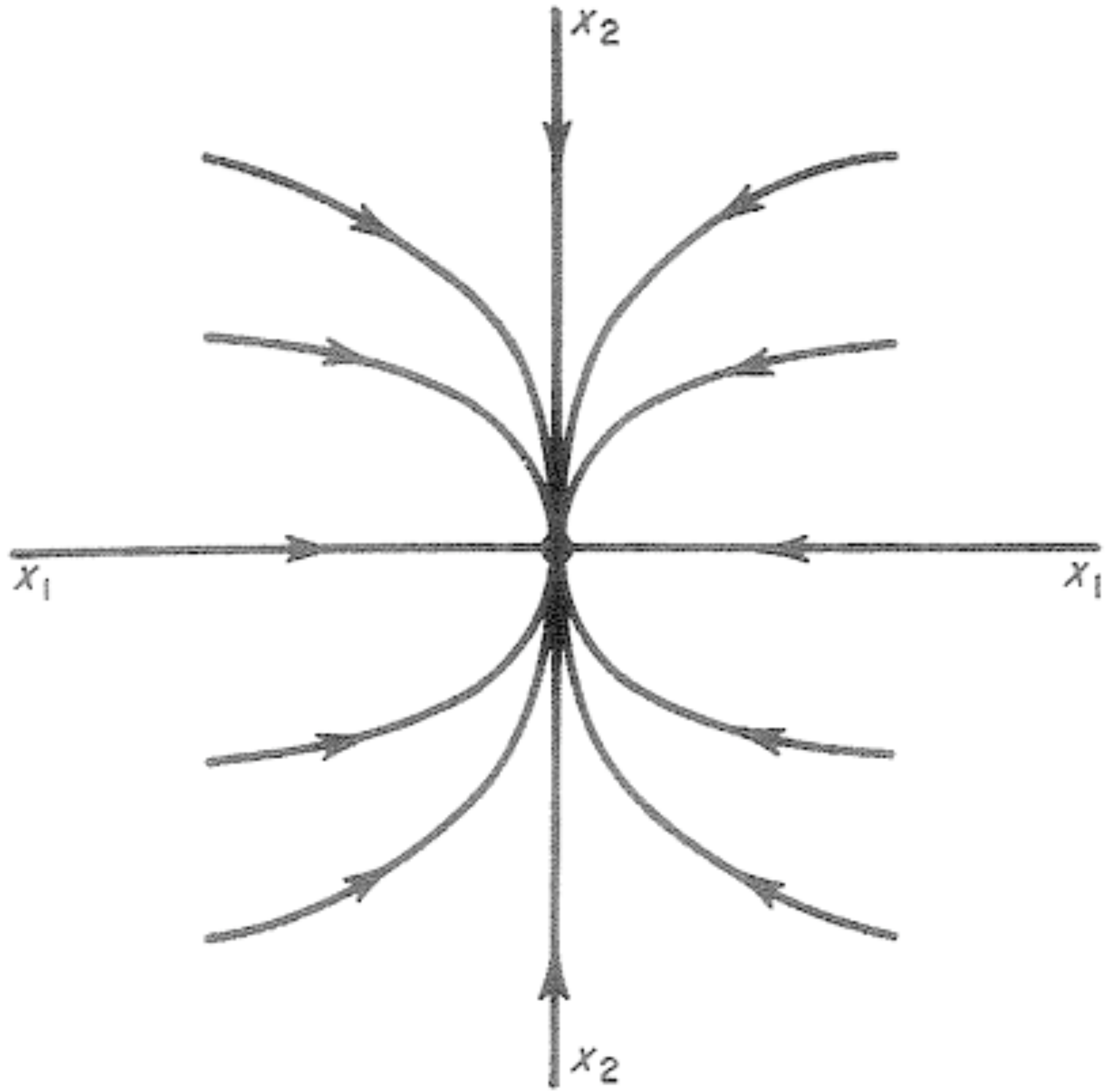


FIG. C. Node: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, $\lambda < \mu < 0$.

Focus Behavior

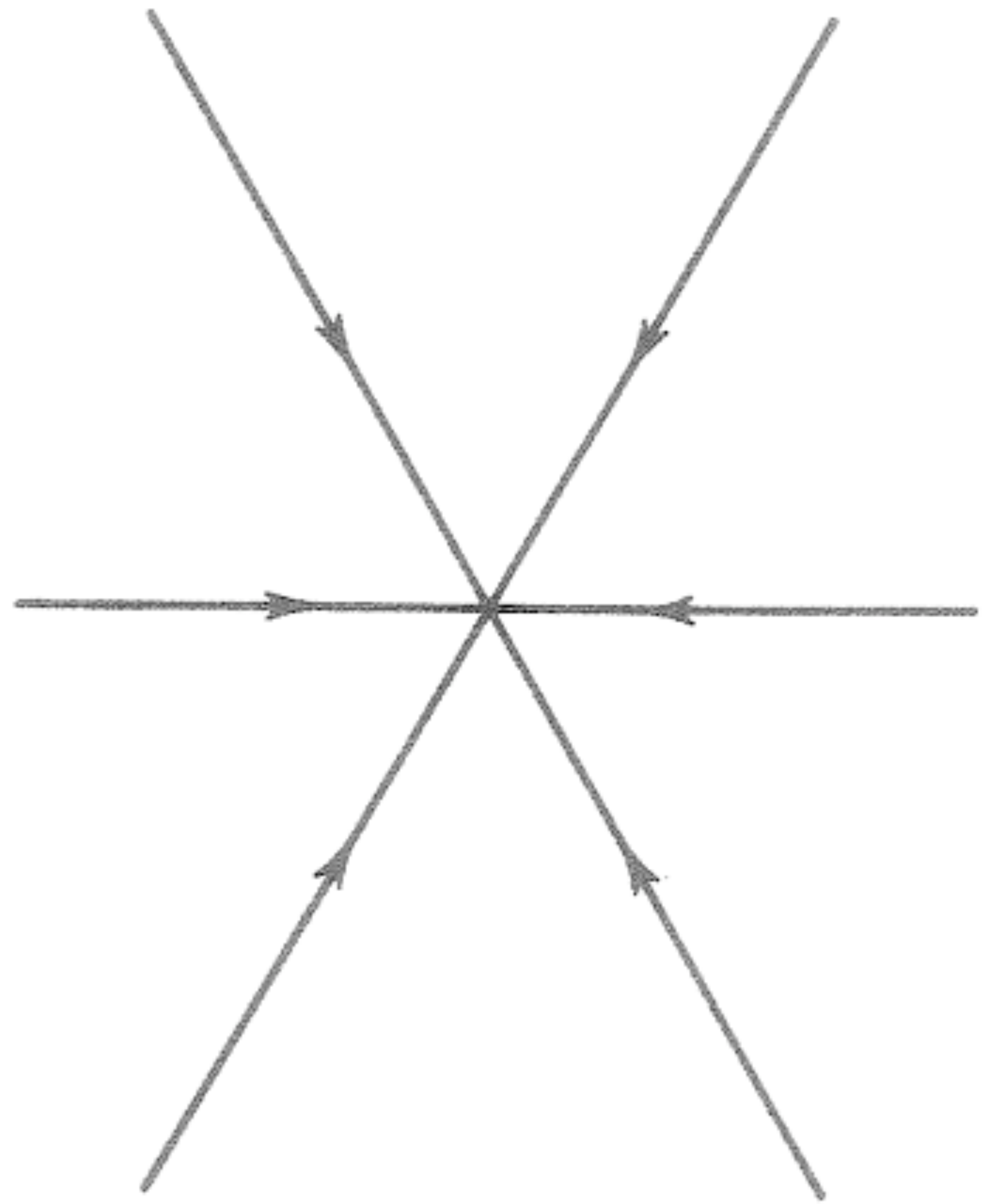


FIG. B. Focus: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $\lambda < 0$.

Saddle Behavior

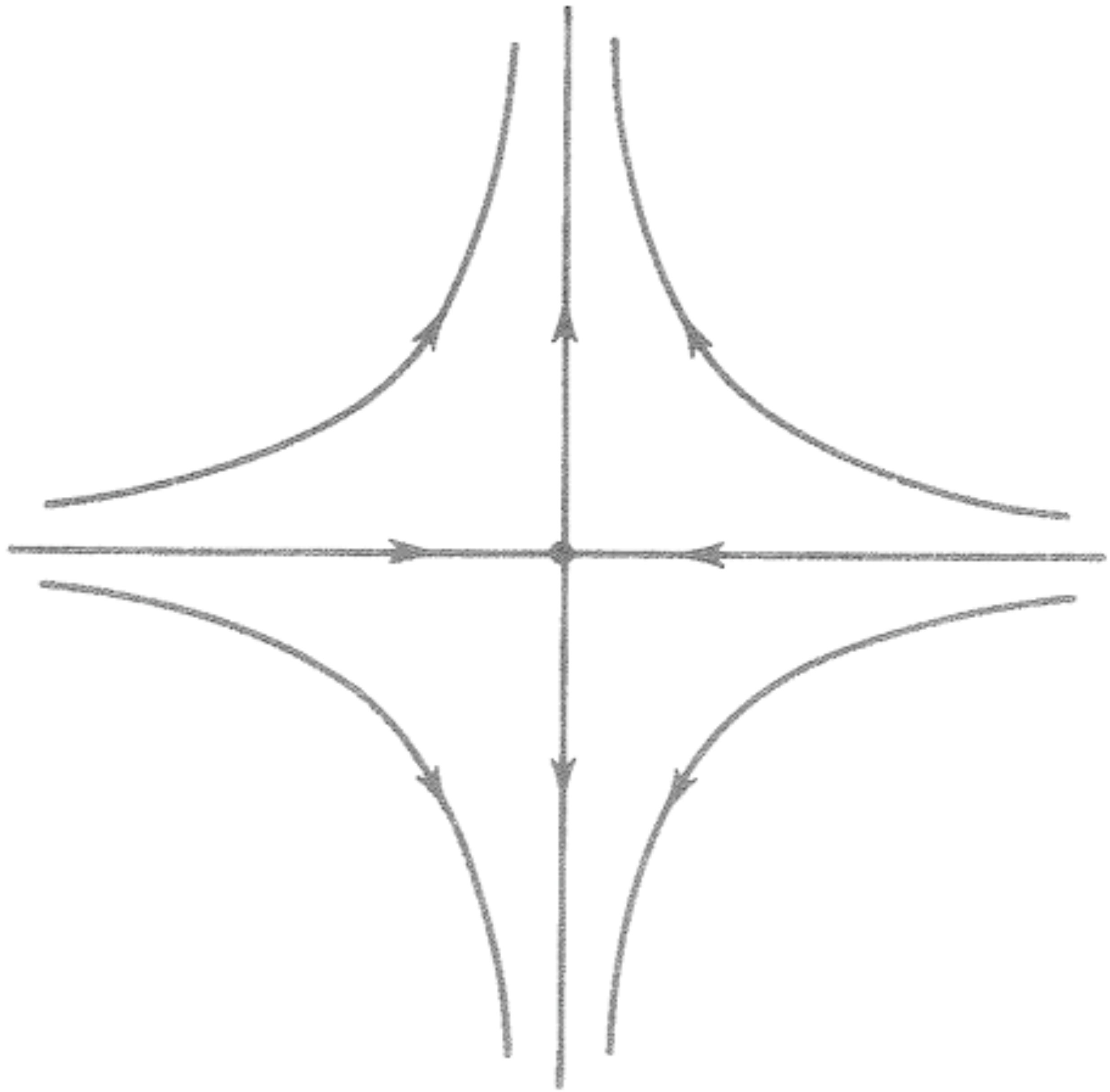


FIG. A. Saddle: $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, $\lambda < 0 < \mu$.

Spiral Behavior

(stable attractor)

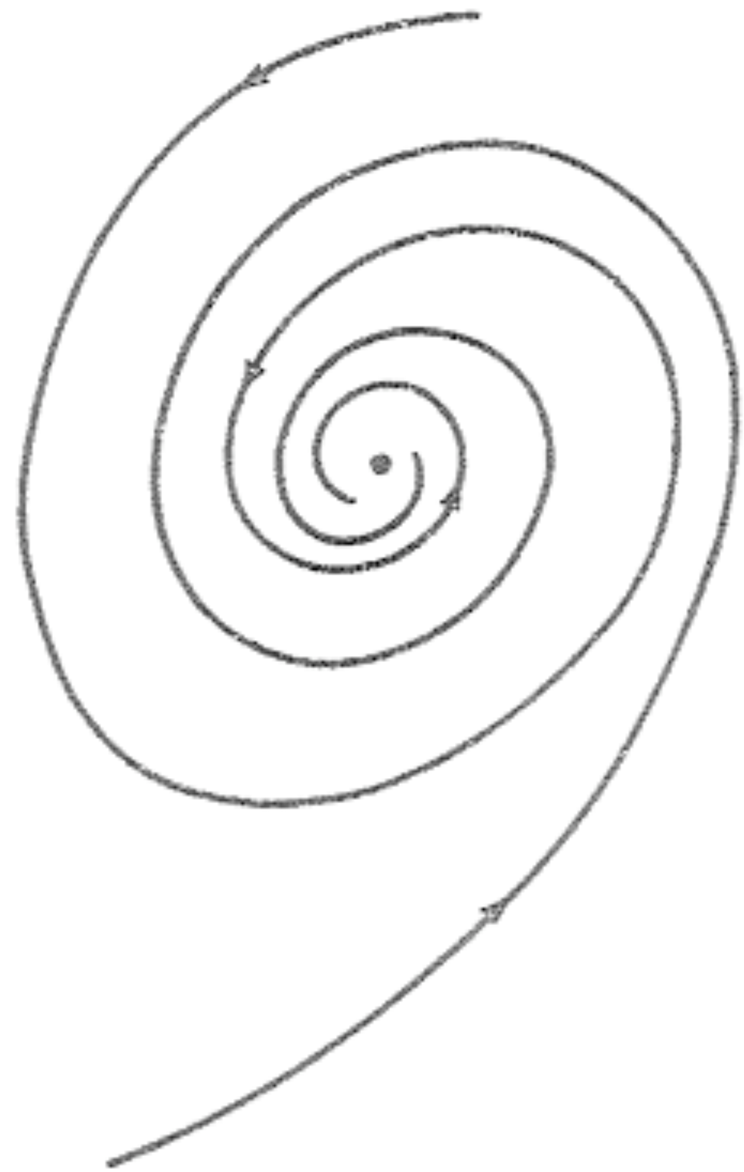


FIG. E. Spiral sink: $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $b > 0 > a$.

Center Behavior

(undamped oscillator)

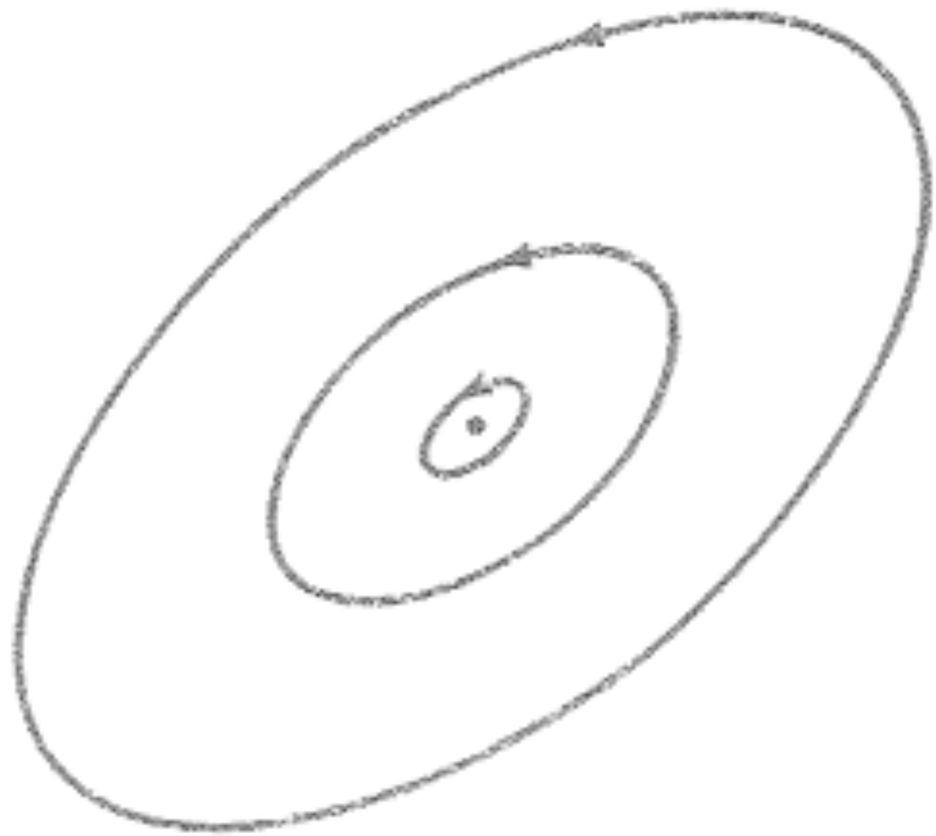
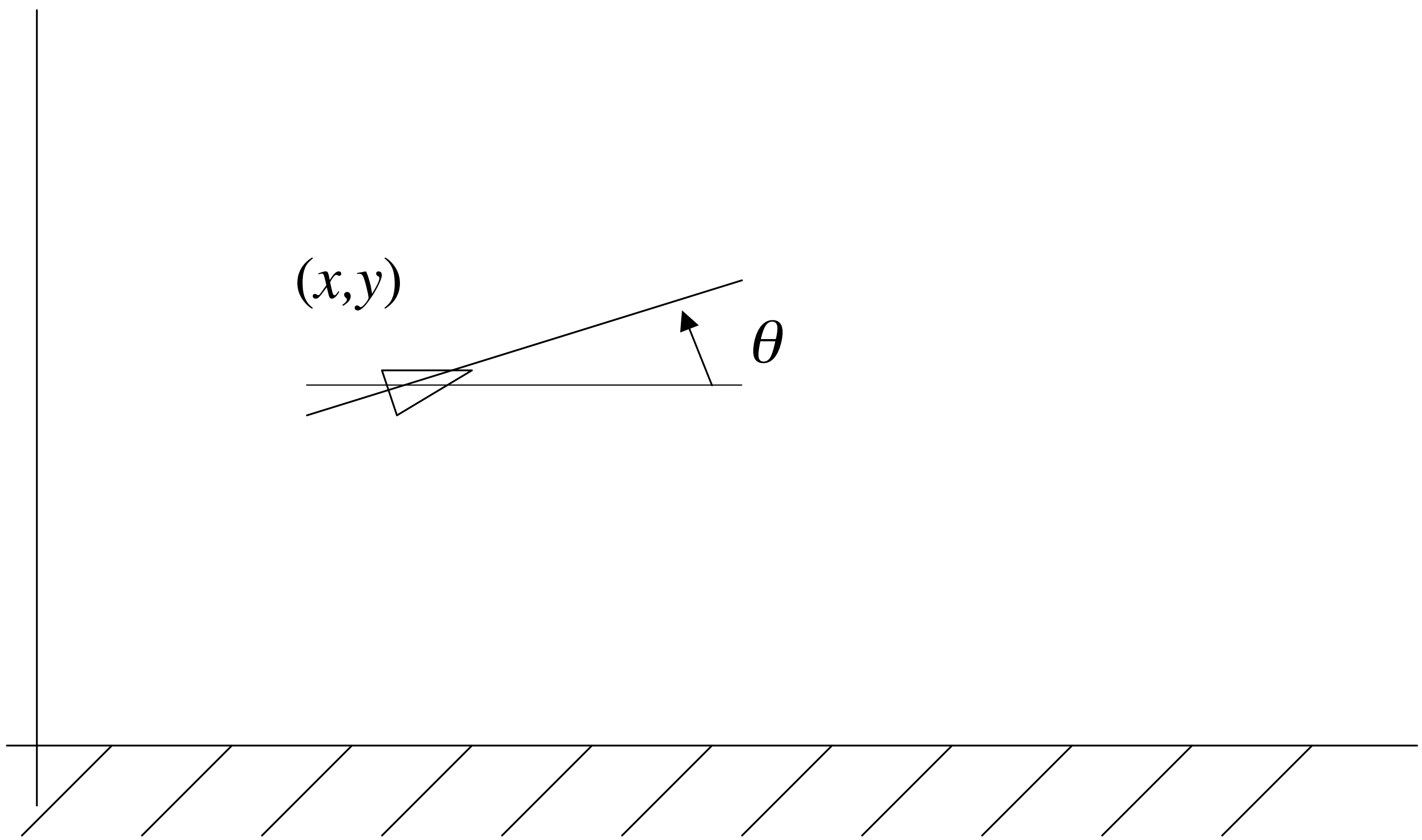


FIG. F. Center: $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}, b > 0.$

The Wall Follower



The Wall Follower

- Our robot model:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = F(\mathbf{x}, \mathbf{u}) = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{pmatrix}$$

$$\mathbf{u} = (v \quad \omega)^{\mathrm{T}} \qquad \mathbf{y} = (y \quad \theta)^{\mathrm{T}} \qquad \theta \approx 0.$$

- We set the control law $\mathbf{u} = (v \quad \omega)^{\mathrm{T}} = H_i(\mathbf{y})$

$$e = y - y_{set} \quad \text{so} \quad \dot{e} = \dot{y} \quad \text{and} \quad \ddot{e} = \ddot{y}$$

The Wall Follower

- Assume constant forward velocity $v = v_0$
  - approximately parallel to the wall: $\theta \approx 0$.
- Desired distance from wall defines error:

$$e = y - y_{set} \quad \text{so} \quad \dot{e} = \dot{y} \quad \text{and} \quad \ddot{e} = \ddot{y}$$

- We set the control law $\mathbf{u} = (v \;\; \omega)^T = H_i(\mathbf{y})$
  - We want e to act like a "damped spring"

$$\ddot{e} + k_1 \dot{e} + k_2 e = 0$$

The Wall Follower

- We want $\ddot{e} + k_1 \dot{e} + k_2 e = 0$
- For small values of θ

$$\begin{aligned} \dot{e} &= \dot{y} = v \sin\theta \approx v\theta \\ \ddot{e} &= \ddot{y} = v \cos\theta \dot{\theta} \approx v\omega \end{aligned}$$

- Assume $v = v_0$ is constant. Solve for ω

$$\mathbf{u} = \begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} v_0 \\ -k_1\theta - \dfrac{k_2}{v_0} e \end{pmatrix} = H_i(e, \theta)$$

– This makes the wall-follower a **PD** controller.

Tuning the Wall Follower

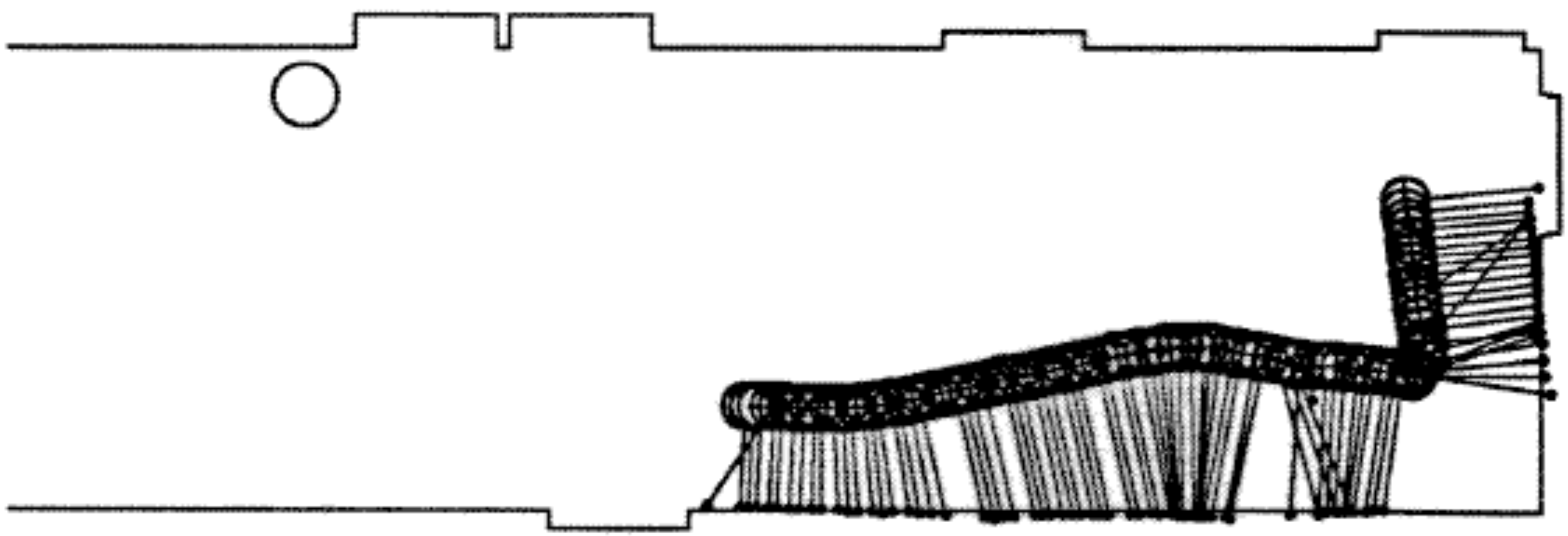
- The system is $\ddot{e} + k_1 \dot{e} + k_2 e = 0$
- Critically damped is $k_1^2 - 4k_2 = 0$

$$k_1 = \sqrt{4 k_2}$$

- Slightly underdamped performs better.
  - Set k_2 by experience.
  - Set k_1 a bit less than $\sqrt{4 k_2}$

An Observer for Distance to Wall

- Short sonar returns are reliable.
  - They are likely to be perpendicular reflections.



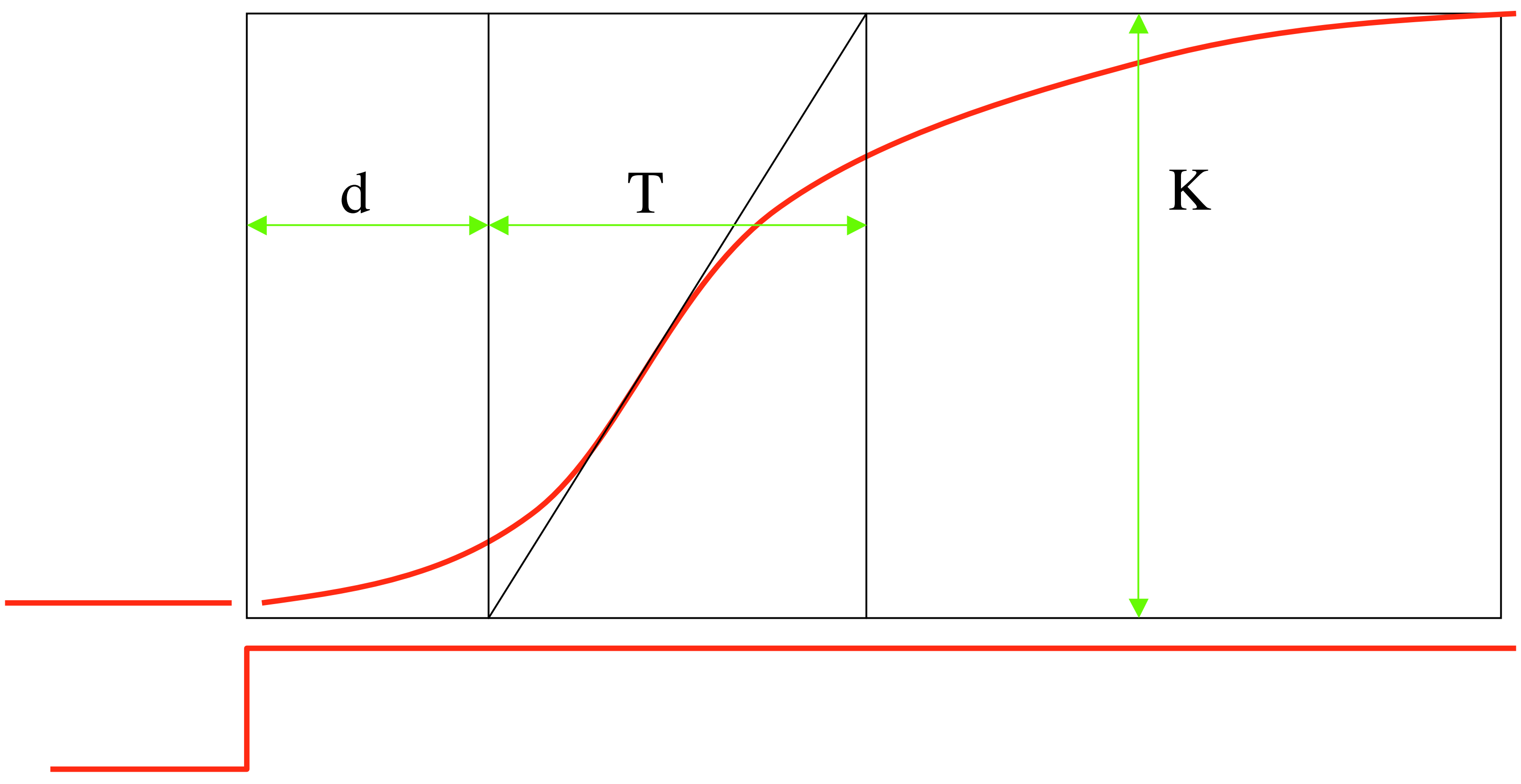
Experiment with Alternatives

- The wall follower is a PD control law.
- A target seeker should probably be a PI control law, to adapt to motion.

- Try different tuning values for parameters.
  - This is a simple model.
  - Unmodeled effects might be significant.

Ziegler-Nichols Tuning

- Open-loop response to a step increase.



Ziegler-Nichols Parameters

- K is the *process gain*.
- T is the *process time constant*.
- d is the *deadtime*.

Tuning the PID Controller

- We have described it as:

$$u(t) = -k_P\, e(t) - k_I \int_0^t e\, dt - k_D \dot{e}(t)$$

- Another standard form is:

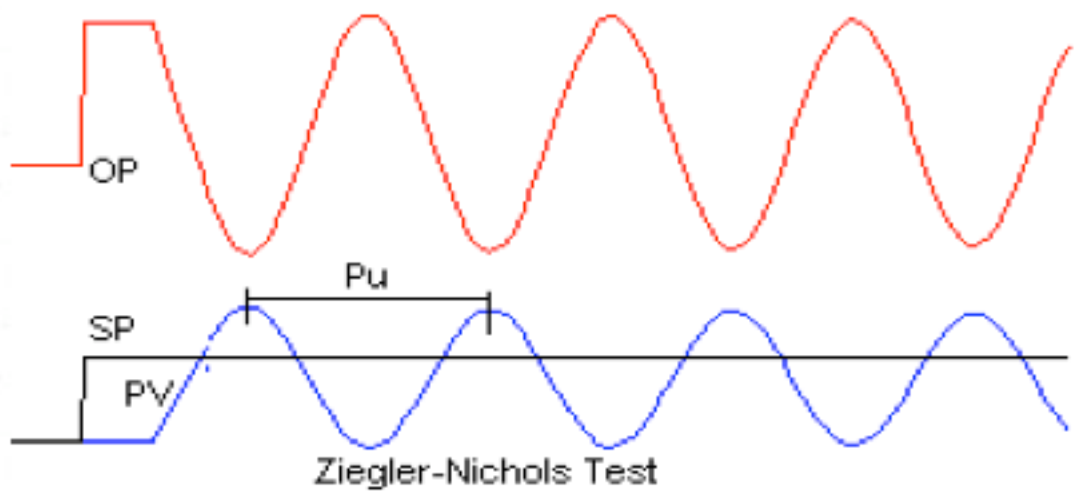
$$u(t) = -P\left[e(t) + T_I \int_0^t e\, dt + T_D \dot{e}(t)\right]$$

- Ziegler-Nichols says:

$$P = \frac{1.5 \cdot T}{K \cdot d} \qquad T_I = 2.5 \cdot d \qquad T_D = 0.4 \cdot d$$

Ziegler-Nichols Closed Loop

1. Disable D and I action (pure P control).
2. Make a step change to the setpoint.
3. Repeat, adjusting controller gain until achieving a stable oscillation.
   - This gain is the "ultimate gain" K_u.
   - The period is the "ultimate period" P_u.



Closed-Loop Z-N PID Tuning

- A standard form of PID is:

$$u(t) = -P\left[e(t) + T_I \int_0^t e\,dt + T_D \dot{e}(t)\right]$$

- For a PI controller:

$$P = 0.45 \cdot K_u \qquad T_I = \frac{P_u}{1.2}$$

- For a PID controller:

$$P = 0.6 \cdot K_u \qquad T_I = \frac{P_u}{2} \qquad T_D = \frac{P_u}{8}$$