Chapter 5 - Vector Spaces: Theory and Practice

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Practical Linear Algebra – Fall 2009

Vector Spaces

Examples of vector spaces:

- $\mathbb{R}$:
- $\mathbb{R}^2$:
- $\mathbb{R}^3$:
- $\mathbb{R}^n$: Set of vectors of the form $x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$, where $\chi_0, \chi_1, \ldots, \chi_{n-1} \in \mathbb{R}$.
  - Direction (vector) from the origin (the point $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$) to the point $x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$.
  - A direction is position independent: You can think of it as a direction anchored anywhere in $\mathbb{R}^n$.

Definition

- Let S be a set.
- This set is called a space if
  - There is a notion of multiplying an element in the set by a scalar: if $x \in S$ and $\alpha \in \mathbb{R}$ then αx is defined.
  - Scaling an element in the set always results in an element of that set: if $x \in S$ and $\alpha \in \mathbb{R}$ then $\alpha x \in S$.
  - There is a notion of adding elements in the set by a scalar: if $x, y \in S$ then $x + y$ is defined.
  - Adding two elements in the set always results in an element of that set: if $x, y \in S$ then $x + y \in S$.

Note

A space has to have the notion of a zero element.

Example

Let $x, y \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

- $z = x + y \in \mathbb{R}^2$;
- $\alpha \cdot x = \alpha x \in \mathbb{R}^2$; and
- $0 \in \mathbb{R}^2$ and $0 \cdot x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Vector spaces

We will talk about *vector spaces* because the spaces have vectors as their elements.

Example

- Consider the set of all real valued $m \times n$ matrices, $\mathbb{R}^{m \times n}$.
- Together with matrix addition and multiplication by a scalar, this set is a vector space.
- Note that an easy way to visualize this is to take the matrix and view it as a vector of length $m \cdot n$.

Example

- Not all spaces are vector spaces.
- The spaces of all functions defined from $\mathbb{R}$ to $\mathbb{R}$ has addition and multiplication by a scalar defined on it, but it is not a vectors space.
- It is a space of functions instead.

Subspaces

- Recall the concept of a subset, B, of a given set, A.
- All elements in B are elements in A.
- If A is a vector space we can ask ourselves the question of when B is also a vector space.

Answer

The answer is that B is a vector space if

- $x, y \in B$ implies that $x + y \in B$
- $x \in B$ and $\alpha \in B$ implies $\alpha x \in B$
- $0 \in B$ (the zero vector).

We call a subset of a vector space that is also a vector space a *subspace*.

Definition

Let A be a vector space and let B be a subset of A. Then B is a subspace of A if

- $x, y \in B$ implies that $x + y \in B$.
- $x \in B$ and $\alpha \in R$ implies that $\alpha x \in B$.

Note

One way to describe a subspace is that it is a subset that is *closed* under addition and scalar multiplication.

Example

The empty set is a subset of $\mathbb{R}^n$. Is it a subspace? Why?

Exercise

What is the smallest subspace of $\mathbb{R}^n$? (Smallest in terms of the number of elements in the subspace.)

Why Should We Care?

Example

Consider

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}$$

- Does $Ax = b_0$ have a solution?

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$$A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}, \quad \text{and} \quad b_1 = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}$$

- Does $Ax = b_0$ have a solution?
- The answer is yes: $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

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- Does $Ax = b_0$ have a solution?
- The answer is yes: $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.
- Does $Ax = b_1$ have a solution? The answer is no.
- Does $Ax = b_0$ have any other solutions?

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- Does $Ax = b_0$ have any other solutions? The answer is yes.

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- Does $Ax = b_0$ have a solution?
- The answer is yes: $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.
- Does $Ax = b_1$ have a solution? The answer is no.
- Does $Ax = b_0$ have any other solutions? The answer is yes.
- How do we characterize all solutions?

Example

Consider the points $(1, 3)$, $(2, -2)$, $(4, 1)$.

- Is there a straight line through these points?
- Under what circumstances is there a straight line through these points?

Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $Ax = b$. Partition

$$A \rightarrow \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \quad \text{and} \quad x \rightarrow \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{array} \right).$$

Then

$$\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1} = b.$$

Definition

Let $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^m$ and $\{\chi_0, \ldots, \chi_{n-1}\} \subset \mathbb{R}$. Then

$$\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}$$

is said to be a *linear combination* of the vectors $\{a_0, \ldots, a_{n-1}\}$.

Note

- Consider $Ax = b$.
- Solution x exists if and only if b equals a linear combination of the columns of A.
- This observation answers the question "Given a matrix A, for what right-hand side vector, b, does $Ax = b$ have a solution?"
- The answer is that there is a solution if and only if b is a linear combination of the columns (column vectors) of A.

Definition

The column space of $A \in \mathbb{R}^{m \times n}$ is the set of all vectors $b \in \mathbb{R}^m$ for which there exists a vector $x \in \mathbb{R}^n$ such that $Ax = b$.

Theorem

The column space of $A \in \mathbb{R}^{m \times n}$ is a subspace (of $\mathbb{R}^m$).

Proof

- The column space of A is closed under addition:
  - Let $b_0, b_1 \in \mathbb{R}^m$ be in the column space of A.
  - Then there exist $x_0, x_1 \in \mathbb{R}^n$ such that $Ax_0 = b_0$ and $Ax_1 = b_1$.
  - But then $A(x_0 + x_1) = Ax_0 + Ax_1 = b_0 + b_1$ and thus $b_0 + b_1$ is in the column space of A.
- The column space of A is closed under scalar multiplication:
  - Let b be in the column space of A and $\alpha \in \mathbb{R}$.
  - Then there exists a vector x such that $Ax = b$.
  - Hence $\alpha Ax = \alpha b$.
  - Since $A(\alpha x) = \alpha Ax = \alpha b$ we conclude that αb is in the column space of A.

Hence the column space of A is a subspace (of $\mathbb{R}^m$).

Example

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}.$$

- Set two appended systems:

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 8 \\ 1 & 2 & 0 & -1 \\ 4 & 1 & 2 & 7 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 1 & 2 & 0 \end{array}\right). \quad (1)$$

- It becomes convenient to swap the first and second equation:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 3 & -1 & 2 & 8 \\ 4 & 1 & 2 & 7 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 3 & -1 & 2 & 0 \\ 4 & 1 & 2 & 0 \end{array}\right).$$

Continued

$$\left(\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 3 & -1 & 2 & 8 \\ 4 & 1 & 2 & 7 \end{array}\right) \qquad \left(\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 3 & -1 & 2 & 0 \\ 4 & 1 & 2 & 0 \end{array}\right).$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & -7 & 2 & 11 \\ 0 & -7 & 2 & 11 \end{array}\right) \qquad \left(\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & -7 & 2 & 0 \end{array}\right).$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & -7 & 2 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & 1 & -2/7 & -11/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 0 & 1 & -2/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$$\left(\begin{array}{rrr|r} 1 & 0 & 4/7 & 15/7 \\ 0 & 1 & -2/7 & -11/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{rrr|r} 1 & 0 & 4/7 & 0 \\ 0 & 1 & -2/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$$\left(\begin{array}{ccr|r} 1 & 0 & 4/7 & 15/7 \\ 0 & 1 & -2/7 & -11/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Bigg| \quad \left(\begin{array}{ccr|r} 1 & 0 & 4/7 & 0 \\ 0 & 1 & -2/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- $0 \times \chi_2 = 0$: no constraint on variable χ_2.
- χ_2 is a **free variable**.
- $\chi_1 - 2/7\chi_2 = -11/7$, or,

  $$\chi_1 = -11/7 + 2/7\chi_2$$

  The value of χ_1 is constrained by the value given to χ_2.
- $\chi_0 + 4/7\chi_2 = 15/7$, or,

  $$\chi_0 = 15/7 - 4/7\chi_2.$$

  Thus, the value of χ_0 is constrained by the value given to χ_2.

Example (continued)

$$\left(\begin{array}{rrr|r} 1 & 0 & 4/7 & 15/7 \\ 0 & 1 & -2/7 & -11/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Bigg| \quad \left(\begin{array}{rrr|r} 1 & 0 & 4/7 & 0 \\ 0 & 1 & -2/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

- $\left(\begin{array}{r} 15/7 - 4/7\chi_2 \\ -11/7 + 2/7\chi_2 \\ \chi_2 \end{array}\right)$ solves the linear system.
- We can rewrite this as

$$\left(\begin{array}{r} 15/7 \\ -11/7 \\ 0 \end{array}\right) + \chi_2 \left(\begin{array}{r} -4/7 \\ 2/7 \\ 1 \end{array}\right).$$

- So, for each choice of χ_2, we get a solution to the linear system.

Note

- $$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 15/7 \\ -11/7 \\ 0 \end{pmatrix}}_{x_p} = \begin{pmatrix} 8 \\ -1 \\ 7 \end{pmatrix}.$$
- x_p is a *particular* solution to $Ax = b_0$. (Hence the p in the x_p.)
- Note that $$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 2 & 0 \\ 4 & 1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} -4/7 \\ 2/7 \\ 1 \end{pmatrix}}_{x_n} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
- For *any* α, $(x_p + \alpha x_n)$ is a solution to $Ax = b_0$:

$$A(x_p + \alpha x_n) = Ax_p + A(\alpha x_n) = Ax_p + \alpha A x_n = b_0 + \alpha \times 0 = b_0.$$

Example (continued)

- The system $Ax = b_0$ has **an infinite number of** solutions.
- To characterize all solutions, it suffices to find one (nonunique) particular solution x_p that satisfies $Ax_p = b_0$.
- Now, for any vector x_n that has the property that $Ax_n = 0$, we know that $x_p + x_n$ is also a solution.

Definition

Let $A \in \mathbb{R}^{m \times n}$. Then the set of all vectors $x \in \mathbb{R}^n$ that have the property that $Ax = 0$ is called the *null space* of A and is denoted by $\mathcal{N}(A)$.

Theorem

The null space of $A \in \mathbb{R}^{m \times n}$ is indeed a subspace of $\mathbb{R}^n$.

Proof

- Clearly $\mathcal{N}(A)$ is a subset of $\mathbb{R}^n$.
- Show that it is closed under addition:
  - Assume that $x, y \in \mathcal{N}(A)$.
  - Then $A(x + y) = Ax + Ay = 0$
  - Therefore $(x + y) \in \mathcal{N}(A)$.
- Show that it is closed under addition:
  - Assume that $x \in \mathcal{N}(A)$ and $\alpha \in \mathbb{R}$.
  - Then $A(\alpha x) = \alpha Ax = \alpha \times 0 = 0$
  - Therefore $\alpha x \in \mathcal{N}(A)$.
- Hence, $\mathcal{N}(A)$ is a subspace.

The zero vector (of appropriate length) is always in the null space of a matrix A.

Systematic Steps to Finding All Solutions

Example

Consider

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Reduction to Row-Echelon Form

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Identify the pivots

The pivots are the first nonzero elements in each row to the left of the line:

$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Give the general solution to the problem

$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

- Identify the free variablese (the variables corresponding to the columns that do not have pivots in them):
  - y and t
- A general solution can be given by

$$\underbrace{\begin{pmatrix} \Box \\ 0 \\ \Box \\ 0 \end{pmatrix}}_{x_p} + \alpha \underbrace{\begin{pmatrix} \Box \\ 1 \\ \Box \\ 0 \end{pmatrix}}_{x_{n_0}} + \beta \underbrace{\begin{pmatrix} \Box \\ 0 \\ \Box \\ 1 \end{pmatrix}}_{x_{n_1}}.$$

Compute x_p

$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

- $x_p = \begin{pmatrix} \square \\ 0 \\ \square \\ 0 \end{pmatrix}$ is a particular (special) solution.
- Set the free variables to zero and solve:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{array}{rcl} x \quad +z & = & 1 \\ 2z & = & 1 \end{array}$$

- Solving this yields $x_p = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}$.

Compute x_{n_0}

$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

- $x_{n_0} = \begin{pmatrix} \square \\ 1 \\ \square \\ 0 \end{pmatrix}$ is the form of one vector in the null space.
- Solve

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ 1 \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ or } \cdots$$

- Solving this yields $x_{n_0} = \begin{pmatrix} \square \\ 1 \\ \square \\ 0 \end{pmatrix}$.

Compute x_{n_1}

$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

- $x_{n_1} = \begin{pmatrix} \square \\ 0 \\ \square \\ 1 \end{pmatrix}$ is the form of another vector in the null space.
- Solve
$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ or } \cdots$$
- Solving this yields $x_{n_0} = \begin{pmatrix} \square \\ 0 \\ \square \\ 1 \end{pmatrix}$.