Cuts from Proofs: A Complete and Practical Technique for Solving Linear Inequalities over Integers

Isil Dillig, Thomas Dillig, and Alex Aiken
Computer Science Department
Stanford University

Linear Arithmetic over Integers

- Problem: Given an $m \times n$ matrix A with only integer entries, and a vector $\vec{b} \in \mathbb{Z}^n$, does

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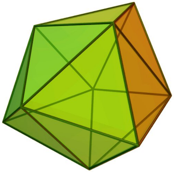
- Geometric interpretation: Are there any integer points inside the polyhedron defined by $A\vec{x} \leq \vec{b}$?



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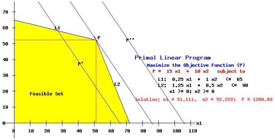
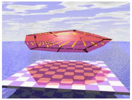
  - Integer overflow checking, RTL-datapath verification, . . .

Existing Techniques

- Simplex-based Approaches:



- The Omega Test:



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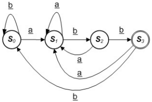

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  - System is satisfiable if the language accepted by the automaton is non-empty.

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This Talk

- A new approach for finding better additional constraints to find an integer solution.
- Performs orders of magnitude better than existing approaches.
- Complete, i.e., guaranteed to find an integer solution if one exists.

Motivating Example

- Consider the system:

$$
\begin{aligned}
-3x + 3y + z &\leq -1 \\
3x - 3y + z &\leq 2 \\
z &= 0
\end{aligned}
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Motivating Example

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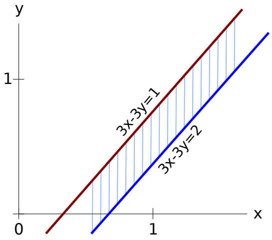
Projection of this system onto the xy plane:



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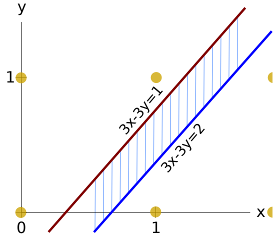

- This system has no integer solutions.

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Example Using Branch and Bound

- For instance, suppose Simplex yields the solution

$$(x, y, z) = \left(\frac{1}{3}, 0, 0\right)$$

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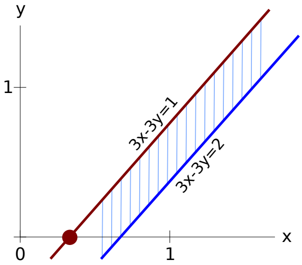

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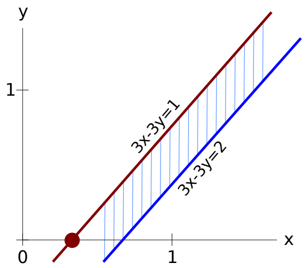
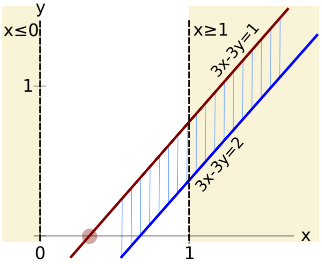


Example Using Branch and Bound

- Branch and bound constructs two subproblems with additional constraints $x \leq 0$ and $x \geq 1$



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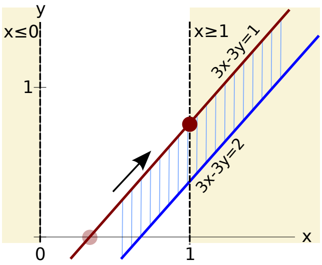
- For the subproblem where $x \geq 1$, we obtain a new solution

$$(x, y, z) = \left(1, \frac{2}{3}, 0\right)$$



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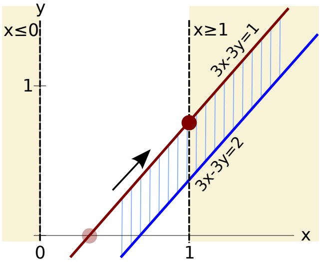
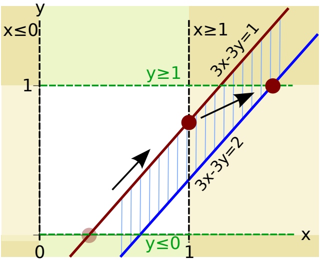


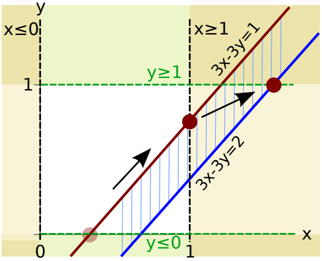
Example Using Branch and Bound

- Now branch and bound constructs another two new subproblems with additional constraints $y \geq 1$ and $y \leq 0$, but the solution is still fractional.



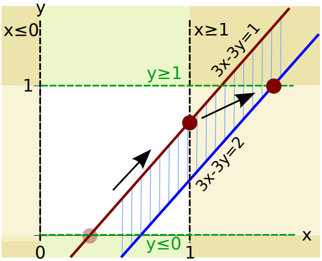
Example Using Branch and Bound

- In fact, by only adding planes parallel to the x and y planes, branch and bound will never exclude the entire space and will keep obtaining more and more fractional solutions.



Example Using Branch and Bound

- While bounds on x and y can be computed to make it terminate, these bounds are extremely large, making branch and bound impractical on its own.



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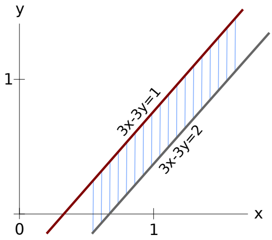

The plane $3x - 3y = 1$ does not contain any integer points.

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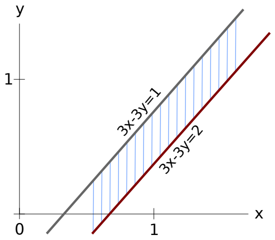

Similarly, $3x - 3y = 2$ also does not contain any integer points.

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Insight

- Instead of excluding individual points on this subspace, we would like to exclude exactly this k-dimensional subspace.

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Insight

- Instead of excluding individual points on this subspace, we would like to exclude exactly this k-dimensional subspace.
- Our technique systematically identifies and excludes these higher dimensional subspaces containing no integer points.

Outline of the Cuts-from-Proofs Algorithm I

Step 1: When Simplex yields a fractional solution, identify the defining constraints of this vertex.

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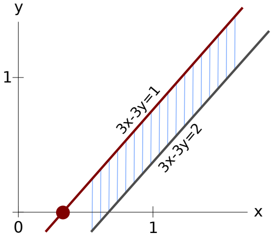
- $-3x + 3y + z \leq -1$ is a defining constraint of $(\frac{1}{3}, 0, 0)$ because $-3 \cdot \frac{1}{3} + 3 \cdot 0 + 0 = -1$.



Outline of the Cuts-from-Proofs Algorithm II

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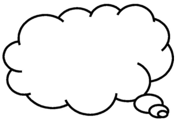
- There may be integer points within the feasible region that lie on this intersection.

Outline of the Cuts-from-Proofs Algorithm III



Idea:

If the intersection of defining constraints does not contain integer solutions, we want to identify the smallest subset of the defining constraints whose intersection does not contain integer solutions.

Smallest subset

$\Rightarrow$

Highest dimensional subspace

Outline of the Cuts-from-Proofs Algorithm IV

Step 3b: If the intersection of defining constraints does not contain an integer point, compute a proof of unsatisfiability and "branch around" this proof.

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- Branching around this proof plane ensures that we exclude at least the intersection of the defining constraints.

- Result: If there is a smaller subset of the defining constraints whose intersection has no integer solution, we will obtain a proof of unsatisfiability for this higher-dimensional intersection in a finite number of steps.

Hermite Normal Forms

Charles Hermite

(1822-1901)

We can determine whether the defining constraints $A'\vec{x} = \vec{b'}$ have an integer solution and also compute proofs of unsatisfiability efficiently (in polynomial time) by using the Hermite Normal Form of A'.

Determining whether Defining Constraints Have Integer Solutions

- Compute H, the Hermite normal form of A', and H^{-1}.

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Important property:

$H^{-1}A'$ is always integral.

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$\Leftrightarrow$

$H^{-1}\vec{b'}$ integral.

Computing Proofs of Unsatisfiability

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Proof of Unsatisfiability

A proof of unsatisfiability of $A'\vec{x} = \vec{b'}$ is:

$$a_1 d_i \cdot x_1 + \ldots + a_n d_i \cdot x_n = n_i$$

Branching around Proofs of Unsatisfiability

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- Then, the closest planes parallel to and on either side of $\Sigma a_i x_i = c_i$ containing integer points are:

$$\Sigma(a_i/g)x_i = \lfloor c_i/g \rfloor \ \text{ and } \ \Sigma(a_i/g)x_i = \lceil c_i/g \rceil$$

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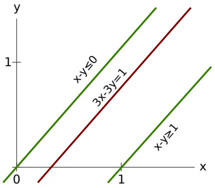
Projection of planes containing integer points on either side of $3x - 3y = 1$



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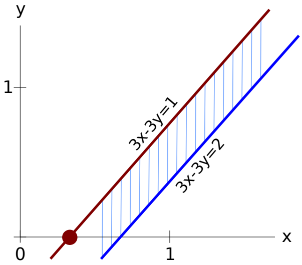
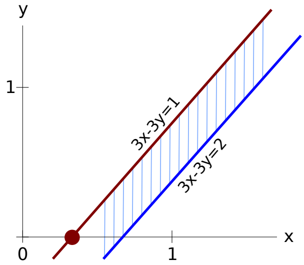


Cuts-from-Proofs Example

The system $A'\vec{x} = \vec{b'}$ is:

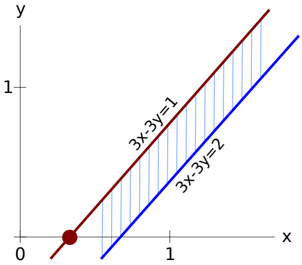
$$\begin{bmatrix} 0 & 0 & 1 \\ -3 & 3 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$



Cuts-from-Proofs Example

Multiply both sides by H^{-1}:

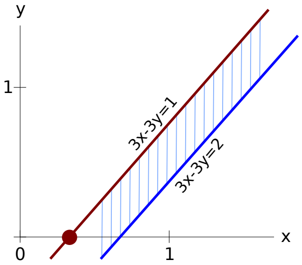
$$\frac{1}{3}\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -3 & 3 & 1 \end{bmatrix}\vec{x} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$



Cuts-from-Proofs Example

Here, $H^{-1}A'x = H^{-1}\vec{b}$ is:

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix}$$



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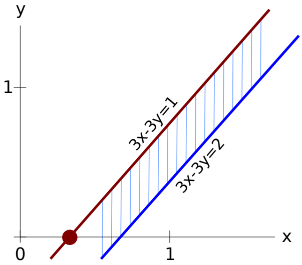
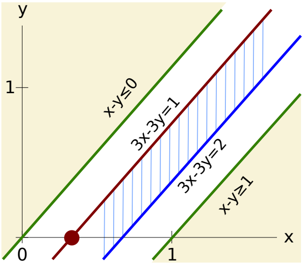
Therefore $-3x + 3y + 3z = -1$ is a proof of unsatisfiability.



Cuts-from-Proofs Example

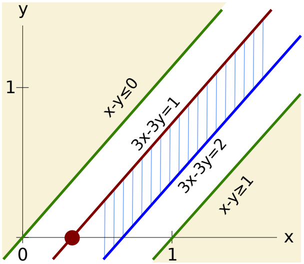
The planes closest to and on either side of the proof plane $-3x + 3y + 3z = -1$ are:

$$\begin{aligned} -x + y + z &= -1 \\ -x + y + z &= 0 \end{aligned}$$



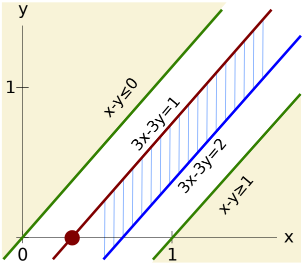
Cuts-from-Proofs Example

Therefore, the Cuts-from-Proofs algorithm solves the two subproblems shown in the figure.



Cuts-from-Proofs Example

Neither subproblem has a real-valued solution, therefore Cuts-from-Proofs terminates in just one step.



Completeness

- To guarantee completeness, it is necessary to restrict the coefficients allowed in the proofs of unsatisfiability to a maximum constant $\alpha \geq n \cdot |a_{max}|$
  - n is the number of variables and $|a_{max}|$ the maximum absolute value of coefficients in the original matrix A.

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  - This is necessary to prevent the volume "cut" by a proof of unsatisfiability from becoming infinitesimally small over time.
  - The constant $n \cdot |a_{max}|$ ensures that if all the proofs of unsatisfiability with coefficients less than or equal to $n \cdot |a_{max}|$ are added, the system will either become infeasible or it contains integer points.

Experiments

- We compare the performance of the Cuts-from-Proofs algorithm against the top four competitors of SMT-COMP'08: Z3, Yices, MathSAT, and CVC3.

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- We did not compare against tools specialized in mixed integer-linear programming, such as CPLEX and GLPK
  - because they do not support infinite precision arithmetic and yield unsound results.

Implementation

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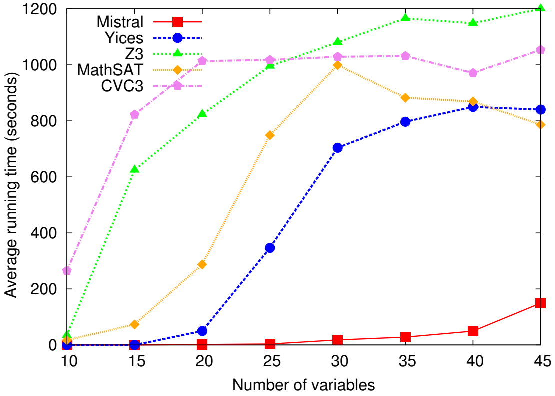
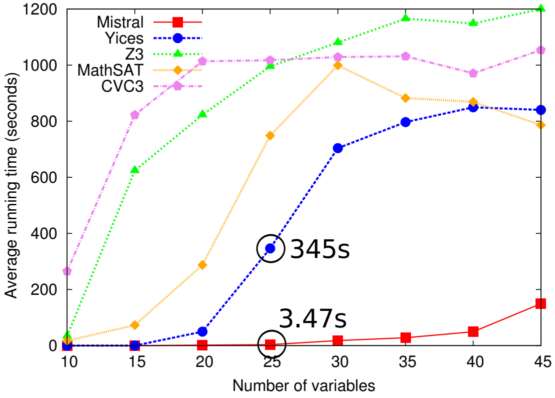
  - But switches to infinite precision representation when overflow is detected.

Experiments



Number of variables vs. average running time. All systems are randomly generated inequalities with fixed coefficient size.

Experiments



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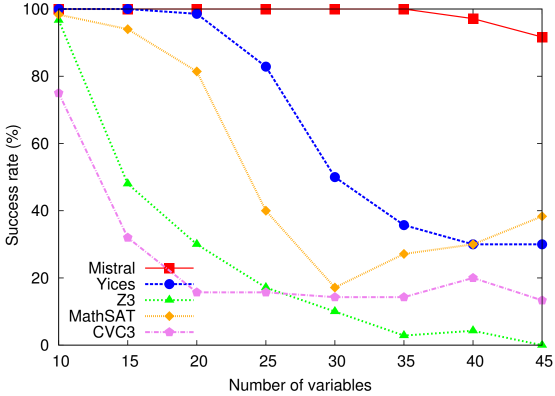

Number of variables vs. percent of successful runs. All systems are randomly generated inequalities with fixed coefficient size.

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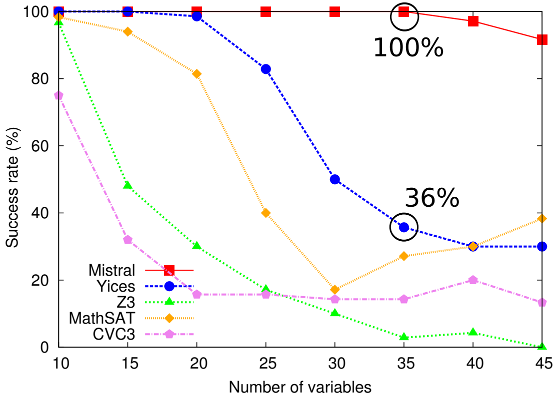

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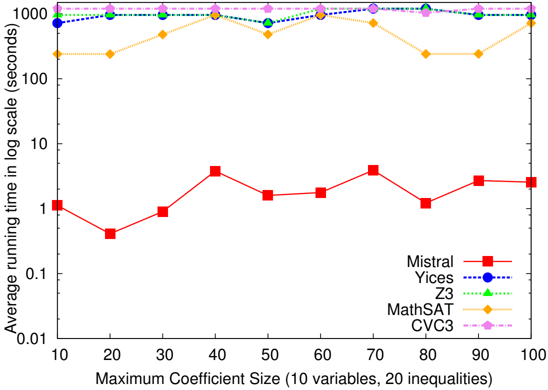

Maximum coefficient vs. average running time for a 10x20 system.

Any Questions?

Related Work

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