CS345H: Programming Languages

Lecture 10: Basic Type Checking

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Outline

- We will write type systems for multiple languages

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- We will formally see how to define soundness

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- We will formally see how to define soundness
- We will learn how to prove soundness of a type system

The let language

- Recall from last time the following small language (let language):

$$
\begin{array}{rcl}
S & \rightarrow & \text{integer} \mid \text{string} \mid \text{identifier} \\
 & & \mid S_1 + S_2 \mid S_1 :: S_2 \\
 & & \mid \text{let } id : \tau \ = \ S_1 \text{ in } S_2 \\
\tau & \rightarrow & Int \mid String
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- Here are again its operational semantics:

$$\frac{\text{integer } i}{E \vdash i : i} \qquad \frac{\text{string } s}{E \vdash s : s} \qquad \frac{\text{identifier } id}{E \vdash id : E(id)} \qquad \frac{\begin{array}{c} E \vdash S_1 : i_1 \\ E \vdash S_2 : i_2 \end{array}}{E \vdash S_1 + S_2 : i_1 + i_2}$$

$$\frac{\begin{array}{c} E \vdash S_1 : s_1 \\ E \vdash S_2 : s_2 \end{array}}{E \vdash S_1 :: S_2 : \text{concat}(s_1, s_2)} \qquad \frac{\begin{array}{c} E \vdash S_1 : e_1 \\ E[\text{id} \leftarrow e_1] \vdash S_2 : e_2 \end{array}}{E \vdash \text{let } id : \tau = S_1 \text{ in } S_2 : e_2}$$

Type System

- We also saw last time how we can write typing rules that compute the type of an expression.

$$\frac{\text{integer } i}{\Gamma \vdash i : Int} \qquad \frac{\text{string } s}{\Gamma \vdash s : String} \qquad \frac{\text{identifier } id}{\Gamma \vdash id : \Gamma(id)}$$

$$\frac{\begin{array}{c}\Gamma \vdash S_1 : Int \\ \Gamma \vdash S_2 : Int\end{array}}{\Gamma \vdash S_1 + S_2 : Int} \qquad \frac{\begin{array}{c}\Gamma \vdash S_1 : String \\ \Gamma \vdash S_2 : String\end{array}}{\Gamma \vdash S_1 :: S_2 : String}$$

$$\frac{\begin{array}{l}\Gamma \vdash S_1 : \tau_1 \\ \tau = \tau_1 \\ \Gamma[\text{id} \leftarrow \tau] \vdash S_2 : \tau_3\end{array}}{\Gamma \vdash \text{let } id : \tau = S_1 \text{ in } S_2 : \tau_3}$$

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  - The concrete environment E corresponds to the abstract type environment Γ
  - The structure of the abstract and concrete rules are analogous
- Key Difference: Concrete semantics compute a particular value, while abstract semantics compute a type

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- Definition: An abstraction is a Galois Connection if $\alpha(\gamma(\tau)) = \tau$ for all abstract values τ
- Question: Is our abstract domain of types a Galois connection? Yes, $\alpha(\gamma(Int)) = Int$ and $\alpha(\gamma(String)) = String$

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- Think of it as a well-formed abstraction
- In this class, we are only interested in Galois connections

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- Slogan: "Well-typed programs cannot go wrong"

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  1. Preservation: Soundness is preserved under transition rules

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  1. Preservation: Soundness is preserved under transition rules
  2. Progress: A well-typed program never "gets stuck" when executing operational semantics (no run-time errors).
- Preservation states that your type system is an overapproximation while progress states that your type system is expressive enough to prevent all run-time errors

How to Prove Preservation

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- This is a very powerful proof technique!

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  Need to prove that $\alpha(i) = Int$
  $\Rightarrow$ This follows directly from the hypothesis that i is an integer

Proving Preservation

- Base case 2:

$$\frac{\text{string } s}{E \vdash s : s} \quad \frac{\text{string } s}{\Gamma \vdash s : String}$$

  Also follows immediately that $\alpha(s) = String$

Proving Preservation

- Inductive case 1:

$$\frac{E \vdash S_1 : i_1 \quad E \vdash S_2 : i_2}{E \vdash S_1 + S_2 : i_1 + i_2} \qquad \frac{\Gamma \vdash S_1 : Int \quad \Gamma \vdash S_2 : Int}{\Gamma \vdash S_1 + S_2 : Int}$$

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- By the inductive hypothesis we know that $\alpha(i_1) = Int$ and $\alpha(i_2) = Int$. Since the value $i_1 + i_2$ is also an integer, $\alpha(i_1 + i_2) = Int$

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- Inductive case 2:

$$\frac{\begin{array}{c} E \vdash S_1 : s_1 \\ E \vdash S_2 : s_2 \end{array}}{E \vdash S_1 :: S_2 : \mathsf{concat}(s_1, s_2)} \qquad \frac{\begin{array}{c} \Gamma \vdash S_1 : String \\ \Gamma \vdash S_2 : String \end{array}}{\Gamma \vdash S_1 :: S_2 : String}$$

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- By the inductive hypothesis we know that $\alpha(s_1) = String$ and $\alpha(s_2) = String$. Since the value $concat(s_1, s_2)$ is also a string, $\alpha(concat(s_1, s_2)) = String$

Proving Preservation with Identifiers

- But what about the two rules involving identifiers?

$$\frac{\text{identifier } id}{E \vdash id : E(id)} \qquad \frac{\text{identifier } id}{\Gamma \vdash id : \Gamma(id)}$$

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- Therefore, we first need to prove agreement before showing the preservation of the identifier rules

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- Base case: E and Γ are empty: $\Rightarrow$ they trivially agree
- Clearly, rules that do not change E or Γ cannot break agreement.
- Therefore, we only have to prove agreement for the following rule:

$$\frac{\begin{array}{l} E \vdash S_1 : e_1 \\ E[\text{id} \leftarrow e_1] \vdash S_2 : e_2 \end{array}}{E \vdash \text{let } id : \tau = S_1 \text{ in } S_2 : e_2} \qquad \frac{\begin{array}{l} \Gamma \vdash S_1 : \tau_1 \\ \tau = \tau_1 \\ \Gamma[\text{id} \leftarrow \tau] \vdash S_2 : \tau_3 \end{array}}{\Gamma \vdash \text{let } id : \tau = S_1 \text{ in } S_2 : \tau_3}$$

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- Here, assuming preservation, we know that $\alpha(e_1) = \tau$. By the inductive hypothesis, we also know that $\Gamma \sim E$.

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$$\frac{E \vdash S_1 : e_1 \qquad E[\text{id} \leftarrow e_1] \vdash S_2 : e_2}{E \vdash \text{let } id : \tau = S_1 \text{ in } S_2 : e_2}$$

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- Here, assuming preservation, we know that $\alpha(e_1) = \tau$. By the inductive hypothesis, we also know that $\Gamma \sim E$.
- Therefore, we also know that $\Gamma[id \leftarrow \tau] \sim E[id \leftarrow e_1]$

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- Here, assuming preservation, we know that $\alpha(e_1) = \tau$. By the inductive hypothesis, we also know that $\Gamma \sim E$.
- Therefore, we also know that $\Gamma[id \leftarrow \tau] \sim E[id \leftarrow e_1]$
- Important: We proved agreement in the inductive case assuming preservation!

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- This follows immediately since by our assumption $\Gamma \sim E$

Proving Preservation with Identifiers cont.

- Inductive case:

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- Observe: We combined agreement and preservation for this proof to work.

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- By the inductive hypothesis, we know that $\alpha(e_2) = \tau_3$. This is what we want to prove.

- Observe: We combined agreement and preservation for this proof to work.
  - The preservation proof works assuming that agreement holds

Proving Preservation with Identifiers cont.

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- As long as both properties hold initially, this is fine!

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- Or more formally, our operational semantics never “get stuck”
- Progress: We need to prove that every program that can be typed under our typing rules will not not “get stuck” in the operational semantics
- Progress is a very strong property that few real type systems obey!

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  - Base case 1: Integer

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  - Base case 2: String

$$\frac{\text{string } s}{E \vdash s : s} \qquad \frac{\text{string } s}{\Gamma \vdash s : String}$$

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  - Base case 2: String

$$\frac{\text{string } s}{E \vdash s : s} \qquad \frac{\text{string } s}{\Gamma \vdash s : String}$$

Clearly, if s types as a string, the corresponding operational semantics rule applies

Proving Progress

- Base case 3: Identifier

$$\frac{\text{identifier } id}{E \vdash id : E(id)} \qquad \frac{\text{identifier } id}{\Gamma \vdash id : \Gamma(id)}$$

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Assuming agreement, we know that if the mapping $id \mapsto \tau$ is present in Γ, the mapping $id \mapsto v$ is present in E. Since this is the only hypothesis (precondition) of the operational semantics rule, it must therefore always apply in all well-types programs

Proving Progress

- Inductive case 1:

$$\frac{\begin{array}{c} E \vdash S_1 : i_1 \\ E \vdash S_2 : i_2 \end{array}}{E \vdash S_1 + S_2 : i_1 + i_2} \qquad \frac{\begin{array}{c} \Gamma \vdash S_1 : Int \\ \Gamma \vdash S_2 : Int \end{array}}{\Gamma \vdash S_1 + S_2 : Int}$$

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$$\frac{E \vdash S_1 : i_1 \quad E \vdash S_2 : i_2}{E \vdash S_1 + S_2 : i_1 + i_2} \qquad \frac{\Gamma \vdash S_1 : Int \quad \Gamma \vdash S_2 : Int}{\Gamma \vdash S_1 + S_2 : Int}$$

We know from the inductive hypothesis that the evaluation of S_1 and S_2 will never get stuck. We also know from preservation that the expressions S_1 and S_2 must evaluate to integers if they are typed Int, therefore the operational semantics rule for plus will always apply since the hypotheses only require that i_1 and i_2 are integers

Proving Progress

- Inductive case 2:

$$\frac{E \vdash S_1 : s_1 \quad E \vdash S_2 : s_2}{E \vdash S_1 :: S_2 : \text{concat}(s_1, s_2)} \qquad \frac{\Gamma \vdash S_1 : String \quad \Gamma \vdash S_2 : String}{\Gamma \vdash S_1 :: S_2 : String}$$

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Proving Progress

- Inductive case 3:

$$\frac{E \vdash S_1 : e_1 \quad E[\mathsf{id} \leftarrow e_1] \vdash S_2 : e_2}{E \vdash \mathsf{let}\ id : \tau = S_1\ \mathsf{in}\ S_2 : e_2} \qquad \frac{\Gamma \vdash S_1 : \tau_1 \quad \tau = \tau_1 \quad \Gamma[\mathsf{id} \leftarrow \tau] \vdash S_2 : \tau_3}{\Gamma \vdash \mathsf{let}\ id : \tau = S_1\ \mathsf{in}\ S_2 : \tau_3}$$

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Here, we know from the inductive hypothesis that $E \vdash S_1 : e_1$ and $E[\text{id} \leftarrow e_1] \vdash S_2 : e_2$ will not get stuck since they are well-typed. Therefore, this rule will also always apply.

Preservation + Progress

- We now proved both preservation and progress of our small type system on the let language.

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- Important Point: You can only prove progress and preservation of a type system with respect to an operational semantics
- Poofs of preservation and progress are always by structural induction
- If you have an environment, you usually need to show agreement to prove preservation
- These proofs tend to always follow the same pattern, so follow this strategy on homeworks/exams

Adding the Lambda to our language

- Let us add the lambda construct to the let-language. I will call this the lambda-language:

$$
\begin{array}{rcl}
S & \rightarrow & \text{integer} \mid \text{string} \mid \text{identifier} \\
 & & \mid S_1 + S_2 \mid S_1 :: S_2 \\
 & & \mid \text{let } id : \tau \; = \; S_1 \text{ in } S_2 \\
 & & \mid \lambda x : \tau . S_1 \\
 & & \mid (S_1 \;\; S_2) \\
\tau & \rightarrow & Int \mid String \mid \tau_1 \rightarrow \tau_2
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$$

- The operational semantics of the new constructs are as follows:

$$
\frac{}{E \vdash \lambda x : \tau . S_1 : \lambda x : \tau \ . S_1}
\qquad
\frac{\begin{array}{l} E \vdash S_1 : \lambda x : \tau . e \\ E \vdash S_2 : e_2 \\ E \vdash e[e_2/x] : e_r \end{array}}{E \vdash (S_1 \ S_2) : e_r}
$$

Typing rules for lambda and Application

- Lambda:

$$\frac{\Gamma[x \leftarrow \tau_1] \vdash S_1 : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . S_1 : \tau_1 \rightarrow \tau_2}$$

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- Observe that these almost exactly correspond to the operational semantics!

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- Observe that these almost exactly correspond to the operational semantics!

- But there is one difference: The body of the let is type checked at the definition, but only evaluated at the application

Preservation for lambda

- Lambda:

$$\frac{}{E \vdash \lambda x : \tau . S_1 : \lambda x : \tau \ . S_1} \quad \frac{\Gamma[x \leftarrow \tau_1] \vdash S_1 : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . S_1 : \tau_1 \rightarrow \tau_2}$$

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- Lambda:

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- First, we observe that if $\Gamma[x \leftarrow \tau_1] \vdash S_1 : \tau_2$ holds, we know by our inductive hypothesis that $\alpha(E \vdash S_1[v/x]) = \tau_2$ for any value v of type τ_1. Therefore, the type of this rule is $\tau_1 \rightarrow \tau_2$

Preservation for Application

- Application:

$$\frac{\begin{array}{l} E \vdash S_1 : \lambda x : \tau . e \\ E \vdash S_2 : e_2 \\ E \vdash e[e_2/x] : e_r \end{array}}{E \vdash (S_1 \ S_2) : e_r} \qquad \frac{\begin{array}{l} \Gamma \vdash S_1 : \tau_1 \rightarrow \tau_2 \\ \Gamma \vdash S_2 : \tau_1 \end{array}}{\Gamma \vdash (S_1 \ S_2) : \tau_2}$$

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- First, we observe by our inductive hypothesis that if the type of S_1 is $\tau_1 \rightarrow \tau_2$, the first hypothesis in the concrete rule must always apply. Second, by the inductive hypothesis we know that $\alpha(e_2) = \tau_1$. Since the type of S_1 is $\tau_1 \rightarrow \tau_2$, we can therefore safely conclude that $\alpha(e_r) = \tau_2$

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- Question: Why could we not formulate the typing rules for lambda and application symmetric to the operational semantics?

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- Answer: Because if we try to type check the body of a lambda at the application site, we have no way of knowing the name of the variable bound in this lambda statement

- This is typical: When typing functions, we usually always examine the function body before it is used

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- But progress is a very useful property, even if it can often only be argued for some classes of run-time errors

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- Next time: Polymorphism