CS345H: Programming Languages

Lecture 12: Type Inference

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- Example: We gave types to let bindings and lambda variables in class
- But annotating types can be cumbersome!
- Anyone who has ever written C++ code can really empathize:

```
vector<Map<int, string> >::const_iterator it...
```

Type Inference

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- Two key points:
  1. Automatically inferring types: This means the programmer has to write no types, but still gets all the benefit from static typing
  2. Inferring the most general type: This means we want to infer polymorphic types whenever possible

Type System

- Here is the type system we used in the lambda language:

$$\frac{\text{integer } i}{\Gamma \vdash i : \mathit{Int}} \qquad \frac{\text{string } s}{\Gamma \vdash s : \mathit{String}} \qquad \frac{\text{identifier } id}{\Gamma \vdash id : \Gamma(id)}$$

$$\frac{\begin{array}{c}\Gamma \vdash S_1 : \mathit{Int} \\ \Gamma \vdash S_2 : \mathit{Int}\end{array}}{\Gamma \vdash S_1 + S_2 : \mathit{Int}} \qquad \frac{\begin{array}{c}\Gamma \vdash S_1 : \mathit{String} \\ \Gamma \vdash S_2 : \mathit{String}\end{array}}{\Gamma \vdash S_1 :: S_2 : \mathit{String}}$$

$$\frac{\begin{array}{l}\Gamma \vdash S_1 : \tau_1 \\ \tau = \tau_1 \\ \Gamma[\text{id} \leftarrow \tau] \vdash S_2 : \tau_3\end{array}}{\Gamma \vdash \text{let } id : \tau = S_1 \text{ in } S_2 : \tau_3}$$

$$\frac{\Gamma[x \leftarrow \tau_1] \vdash S_1 : \tau_2}{\Gamma \vdash \lambda x : \tau_1 . S_1 : \tau_1 \rightarrow \tau_2} \qquad \frac{\begin{array}{l}\Gamma \vdash S_1 : \tau_1 \rightarrow \tau_2 \\ \Gamma \vdash S_2 : \tau_1\end{array}}{\Gamma \vdash (S_1 \ S_2) : \tau_2}$$

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- Therefore, the type of `g` must be $\forall \alpha . Int \rightarrow \alpha$
- This means that the type of `f2` is $\forall \alpha . (Int \rightarrow \alpha) \rightarrow \alpha$

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- For this, let us look at the type derivation for the following simple function:
  `lambda x:Int.x+2`
- Here is the type derivation tree for this expression:

$$\dfrac{\dfrac{\begin{array}{c}\textit{identifer } x \\ \Gamma(x) = \textit{Int}\end{array}}{\Gamma[x \leftarrow \textit{Int}] \vdash x : \textit{Int}} \qquad \dfrac{\textit{integer } 2}{\Gamma[x \leftarrow \textit{Int}] \vdash 2 : \textit{Int}}}{\dfrac{\Gamma[x \leftarrow \textit{Int}] \vdash x + 2 : \textit{Int}}{\Gamma \vdash \lambda x{:}\textit{Int}.x + 2 : \textit{Int} \rightarrow \textit{Int}}}$$

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- Specifically, pretend that the type of the argument is just some type variable called `a`
- And for all rules that have preconditions on `a`, write these preconditions as constraints

Type Variables Cont.

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$$\dfrac{\dfrac{\dfrac{\begin{array}{c}identifer\ x\\ \Gamma(x) = a\end{array}}{\Gamma[x \leftarrow a] \vdash x : a} \qquad {\color{red} a = Int} \qquad \dfrac{integer\ 2}{\Gamma[x \leftarrow a] \vdash 2 : Int}}{\Gamma[x \leftarrow a] \vdash x + 2 : Int}}{\Gamma \vdash \lambda x{:}a.x + 2 : a \rightarrow Int}$$

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  - After we typed the expression, we had a the type $a \rightarrow Int$ and the constraint $a = Int$
  - Solving the type with respect to the collected constraint yields: $Int \rightarrow Int$

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Generalizing our typing rules

- The base cases stay unchanged:

$$\frac{\text{integer } i}{\Gamma \vdash i : Int} \quad \frac{\text{string } s}{\Gamma \vdash s : String} \quad \frac{\text{identifier } id}{\Gamma \vdash id : \Gamma(id)}$$

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- When type checking plus, we now collect constraints on the operands:

$$\frac{\begin{array}{c}\Gamma \vdash S_1 : \tau_1 \\ \Gamma \vdash S_2 : \tau_2 \\ \tau_1 = Int, \tau_2 = Int\end{array}}{\Gamma \vdash S_1 + S_2 : Int}$$

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- Specifically, this rule now succeeds as long as S_1 and S_2 evaluate to any type, we simply collect constraints on the types τ_1 and τ_2 to be processed later

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- Let's move on to the typing rule for let:

$$\frac{\begin{array}{l}\Gamma[\text{id} \leftarrow a] \vdash S_1 : a \quad (a \text{ fresh}) \\ \Gamma[\text{id} \leftarrow a] \vdash S_2 : \tau\end{array}}{\Gamma \vdash \text{let } id = S_1 \text{ in } S_2 : \tau}$$

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- Observe that this case only introduces a type variable, but does not add any constraints

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- Let's move on to the typing rule for lambda:

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- We also use this type variable in the return type

Application

- Now the only rule missing so far is application:

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- In this rule, this type variable encodes the return type of the application

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- Final Type: $a_1 \rightarrow Int$ under constraints $a_1 = Int, Int = Int$

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- Solving this type yields $Int \rightarrow Int$

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- We will omit the quantifier from type variables and assume that any type variable is implicitly universally quantified

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- Observe that step 1 can never get stuck! We now reject all programs that cannot be types in step 2.

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- Example: Consider again the expression `let f = lambda x.(f x) in f`
- Here, the type of `f` written as list of constraints is:

$$a_1 = a_2 \rightarrow a_3$$
$$a_1$$

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- This can be done using unification

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- Here, we pick a_1. It's right-hand side is $a_2 \to a_3$. If we replace all occurrences of a_1, we get:

$$a_2 \to a_3 = a_2 \to a_3$$
$$a_2 \to a_3$$

and the substitution $\sigma = \{a_1 \leftarrow (a_2 \to a_3), a_2 \leftarrow a_2, a_3 \leftarrow a_3\}$

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- In this case, we have found the most general solution.

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- But now we are stuck, even though the final substitution is $\sigma = \{a_2 \leftarrow String, a_3 \leftarrow Int, ...\}$

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with $\sigma = \{a_2 \leftarrow a_2, a_3 \leftarrow a_3\}$

- Add $s_2 \leftarrow Int$ and $a_3 \leftarrow String$
- New constraint system:

$$String \to Int = String \to Int$$

with $\sigma = \{a_2 \leftarrow String, a_3 \leftarrow Int\}$

Simple Unification Algorithm

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- Repeat until we either derive a contradiction or there are not equalities left. This is a most general unifier.

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- But this algorithm is not very efficient.
- Next time: How to perform unification efficiently and type inference in L