Vector and Affine Math II

Linear Transformations

Given vector space V and W, function f: V —> W is a linear map (linear transformation) if

$$f(a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m) = a_1 f(\mathbf{v}_1) + \ldots + a_m f(\mathbf{v}_m)$$

Transformations

A 2D transformation matrix:

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Applied to a 2D vector:

$$\mathbf{v}' = \mathbf{M}\mathbf{v}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In which case:

$$x' = ax + by$$
$$y' = cx + dy$$

Scaling

Suppose b = c = 0, but a and d can take on any positive value…

Scaling matrix: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

What happens if a and d are not equal?

Reflection

Suppose b = c = 0, but either a or d goes negative

Reflection matrices:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Across which axes will each of these matrices reflect?

Shear

Suppose a = d = 1, but b or c changes value

Shear matrix:

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \qquad \begin{aligned} x' &= x + by \\ y' &= y \end{aligned}$$

Skews in one dimension in 2D

What does a shear do in 3D?

In-class Exercises

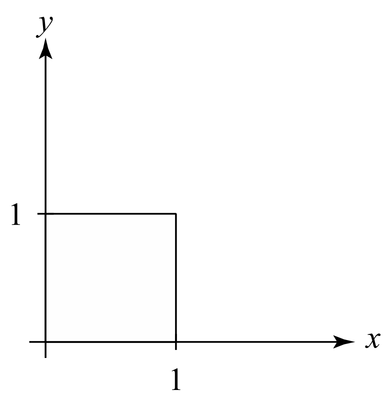
1. Create a 2D box with Euclidean coordinates. Now separately:
    1. Apply a uniform and non-uniform scaling to its vertices
    2. Apply reflection to its vertices
    3. Apply a shear to its vertices
2. Draw all of these transformations

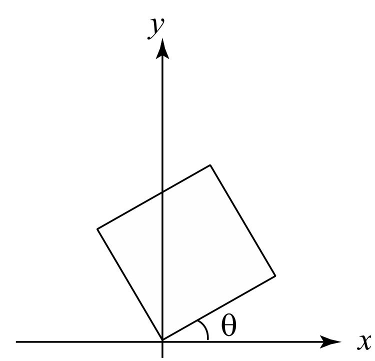
For all activities, show matrices

Rotation

Rotation about the origin:

$$M_R = R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$





$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Linear Transformation Limitations

No notion of an origin

What important graphics operation does this leave out?

Affine Transformations

- Augment linear space **u**, **w** with an origin, **t**
- **u** and **w** are basis vectors
- **t** is a point
- A change of frame looks like:

$$\mathbf{p}' = x \cdot \mathbf{u} + y \cdot \mathbf{w} + \mathbf{t}$$

- How do you represent linear transformations within affine frames?

Homogeneous Coordinates

Loft problem into next dimension:

$$\begin{aligned}
\mathbf{p}' &= \mathbf{M}\mathbf{p} \\
&= \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{u} & \mathbf{w} & \mathbf{t} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\
&= x \cdot \mathbf{u} + y \cdot \mathbf{w} + 1 \cdot \mathbf{t}
\end{aligned}$$

Note that $[a\ c\ 0]^T$ and $[b\ d\ 0]^T$ represent vectors and $[t_x\ t_y\ 1]^T$, $[x\ y\ 1]^T$ and $[x'\ y'\ 1]^T$ represent points.

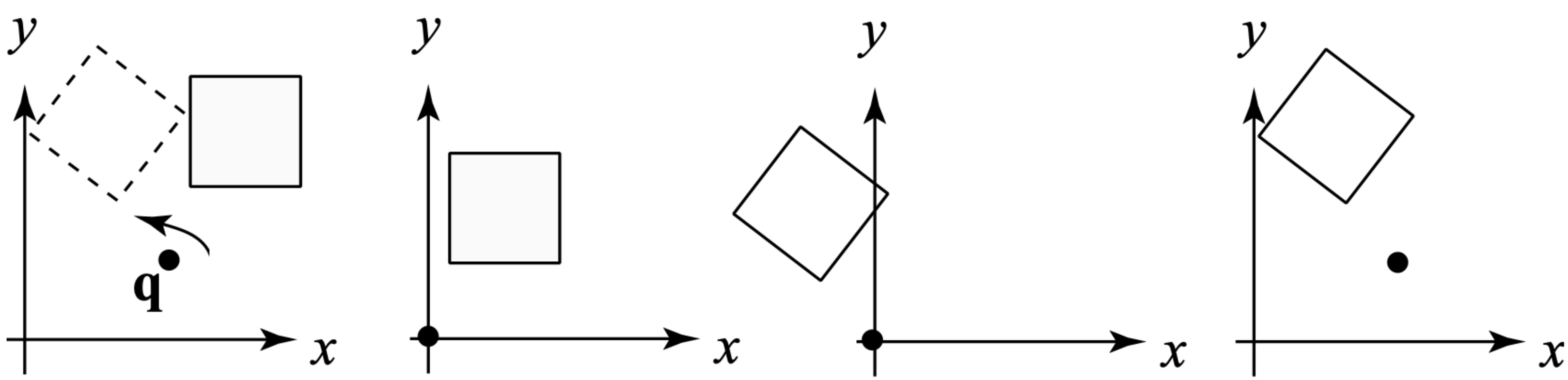
In-class Exercises

1. Create a 2D box with Euclidean coordinates. Now separately:
    1. Apply a uniform and non-uniform scaling to its vertices
    2. Apply reflection to its vertices
    3. Apply a shear to its vertices
    4. Apply a translation then a rotation
    5. Apply a rotation then a translation
2. Draw all of these transformations

For all activities, show matrices

Rotation Around Arbitrary Points

1. Translate q to origin
2. Rotate
3. Translate back



Note that transformation order matters!

Additional Concepts

- Parametric Line Segments
- Plane Equation
- Barycentric Coordinates

All core concepts for working with raytracing! (Assignment 1)

Parametric Line Segment

Linear interpolation along a line, ray or line segment:

$$p(t) = p_0 + t(p_1 - p_0) = (1 - t)p_0 + tp_1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix} = \begin{bmatrix} (1-t)x_0 + tx_1 \\ (1-t)y_0 + ty_1 \\ (1-t)z_0 + tz_1 \end{bmatrix}$$

Line segment: $0 \leq t \leq 1$

Ray: $0 \leq t \leq \infty$

Line: $-\infty \leq t \leq \infty$

Plane Equation

Given normal vector N orthogonal to the plane and any point p' in the plane, p is in plane if: $(p - p') \cdot N = 0$

This can be rewritten: $N \cdot p + d = 0$

Where

$$d = -(N_x p'_x + N_y p'_y + N_z p'_z)$$

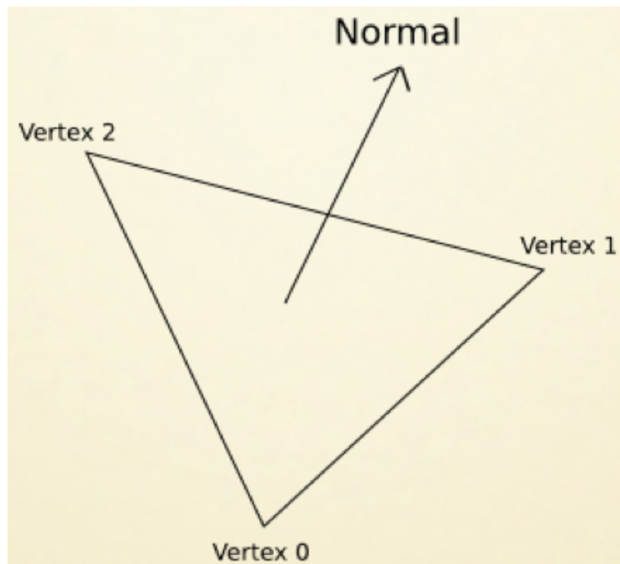
Plane Equation

$$N \cdot p + d = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + d = ax + by + cz + d = 0$$

Triangle Normal

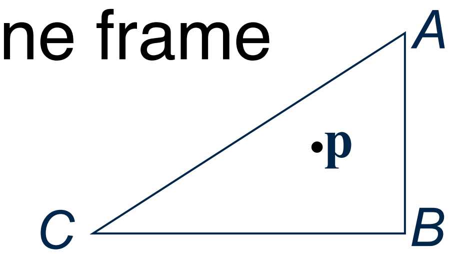
$$N = normalize((v_1 - v_0) \times (v_2 - v_0))$$



Note: Order matters to point the normal in "front-facing" direction (CCW for right-handed systems)

Barycentric Coordinates

A set of points can be used to create an affine frame



Form a frame with an origin C and vectors from C to other vertices: $\mathbf{u} = A - C \quad \mathbf{v} = B - C \quad t = C$

Write p in this coordinate frame: $\mathbf{p} = \alpha\mathbf{u} + \beta\mathbf{v} + \mathbf{t}$

Coordinates (α, β, γ) are called the barycentric coordinates of p relative to A, B, and C

In-class Exercises

1. Create a 2D box with Euclidean coordinates. Now separately:
    1. Apply a uniform and non-uniform scaling to its vertices
    2. Apply reflection to its vertices
    3. Apply a shear to its vertices
    4. Apply a translation then a rotation
    5. Apply a rotation then a translation
2. Draw all of these transformations
3. Find point p' along an edge of this box using the parametric equation and t values 0.1, 0.4 and 0.7
4. Create a point p" within the box and calculate its barycentric coordinates using one of the box points as your origin

For all activities, show matrices