Linear Algebra Worksheet

"There is hardly any theory which is more elementary [than linear algebra], in spite of the fact that generations of professors and textbook writers have obscured its simplicity by preposterous calculations with matrices." –J. Dieudonne

Name: ______________________________

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> Questions begin on the next page. Please answer each question in the spaces provided; if you need extra room, you can attach extra blank pages at the end of the worksheet.
>
> The worksheet follows the standard class collaboration policy. You may ask other students, the instructor, or the TA for help, but **all work written on this worksheet must be entirely your own.** Do not simply copy somebody else's answers.
>
> You may look at linear algebra textbooks, online tutorials, math.stackexchange, or any other resources to help you complete the worksheet. Do not worry if you cannot do the problems from memory alone! You are allowed, and expected, to look up any formulas or definitions as needed.
>
> The problems are not intended to be difficult or time-consuming to solve, but rather, to refresh your memory on basic linear algebra concepts, and to give you a self-check on the prerequisites you will need to succeed in this class. If you get stuck on any problem, the TA or instructor would be happy to help you during office hours.

Useful Formulas

- Dot product: $(u_x, u_y, u_z, u_w) \cdot (v_x, v_y, v_z, v_w) = u_x v_x + u_y v_y + u_z v_z + u_w v_w$.
- Cross product: $(u_x, u_y, u_z) \times (v_x, v_y, v_z) = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x)$.
- Norm: $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\mathbf{u}^T \mathbf{u}}$.
- Unit vector: $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$.
- Angles between vectors: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta, \quad \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\mathbf{u}\|\|\mathbf{v}\|| \sin\theta|$.

1. Recall that a function $f(\mathbf{v})$ of a vector $\mathbf{v}$ is *linear* if $f(\alpha\mathbf{v}) = \alpha f(\mathbf{v})$ and $f(\mathbf{v}+\mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$. For each of the following functions, (a) state whether or not the function is linear, and (b) if the function is not linear, give a counterexample where the function violates one of the above properties.
   (a) (1 point) Rescaling of the input vector, $f(\mathbf{v}) = 5\mathbf{v}$.

   (b) (1 point) The constant function $f(\mathbf{v}) = 2$.

   (c) (1 point) Translation $f(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ for some fixed vector $\mathbf{w} \neq \mathbf{0}$.

   (d) (1 point) The function returning the first coordinate of a vector, $f\left(\begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}\right) = v_0$.

   (e) (2 points) Multiplication by a fixed matrix $M_{n\times m}$, $f(\mathbf{v}) = M\mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^m$.

   (f) (2 points) $f(\mathbf{v}) = \mathbf{w} \cdot \mathbf{v}$, for a fixed vector $\mathbf{w}$.

   (g) (2 points) The norm function $f(\mathbf{v}) = \|\mathbf{v}\|$.

2. Although in practice you will not need to do a lot of linear algebra by hand (that's what computers are for!), you should be able to perform basic computations, and know how to use computational resources like Matlab, Mathematica, Wolfram Alpha to help you debug your code.

   (a) (3 points) Compute

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

   (b) (3 points) Does the matrix

$$M = \begin{bmatrix} \pi & 9 \\ 1 & \pi \end{bmatrix}$$

   have an inverse? How do you know for sure? If an inverse exists, compute M^{-1} exactly.

   (c) (4 points) Use your favorite computer linear algebra tool to solve

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ -1 & 1 & 2 & -3 \\ 3 & -1 & 4 & -1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

   for $\mathbf{v}$. Round each coordinate of your answer to the nearest 0.01.

3. Let $\mathbf{w} \in \mathbb{R}^2$ be a fixed vector. Let

$$f(\mathbf{v}) = \left(I - \frac{\mathbf{w}\mathbf{w}^T}{\|\mathbf{w}\|^2}\right)\mathbf{v},$$

for vectors $\mathbf{v} \in \mathbb{R}^2$ in the plane, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

(a) (2 points) Is this function linear? If not, provide a counterexample.

(b) (2 points) Compute $f(\mathbf{w})$.

(c) (2 points) Compute $f(\mathbf{v})$, where $\mathbf{v}$ is any vector perpendicular to $\mathbf{w}$ (i.e. $\mathbf{v} \cdot \mathbf{w} = 0$).

(d) (4 points) Describe using words, and no equations, what the function f does to vectors. To build intuition, it may help to try concrete numerical examples, or to draw a picture.

4. A key skill in linear algebra is turning systems of linear equations into matrix equations of the form $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are known, and $\mathbf{x}$ is unknown.

   (a) (5 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{w}$ be four different vectors in $\mathbb{R}^3$. You are told that

   $$\mathbf{w} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$$

   for some unknown real numbers α, β, and γ. Write a matrix equation (of the form $A\mathbf{x} = \mathbf{b}$) that would allow you to solve for these unknown scalars.

   (b) (5 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three different vectors in $\mathbb{R}^3$, and k_1, k_2, k_3 three real numbers. You are told that

   $$k_1 = \mathbf{w} \cdot \mathbf{v}_1$$
   $$k_2 = \mathbf{w} \cdot \mathbf{v}_2$$
   $$k_3 = \mathbf{w} \cdot \mathbf{v}_3$$

   for some unknown vector $\mathbf{w} \in \mathbb{R}^3$. Write down a matrix equation that would allow you to solve for this vector $\mathbf{w}$.