Stable Models of Positive Programs, Part 1

From now on, we will assume that bodies of rules contain neither $\rightarrow$ nor $\leftrightarrow$. In other words, bodies of rules will be first-order formulas built from atomic formulas and the 0-place connectives $\top$, $\bot$ using

$$\wedge,\ \vee,\ \neg,\ \forall,\ \exists.$$

A logic program is *positive* if every negation in it is applied to an equality. For instance, the rule

$$Brother(x,y) \leftarrow Parent(z,x) \wedge Parent(z,y) \wedge Male(x) \wedge x \neq y$$

is positive, and the rule

$$Female(x) \leftarrow \neg Male(x) \wedge Person(x)$$

is not.

We will first define the concept of a stable model for positive programs containing a single predicate constant, such as

$$P(a),\ P(b), \tag{1}$$

or

$$\begin{array}{l} P(a,b), \\ P(x,y) \leftarrow P(y,x), \end{array} \tag{2}$$

or

$$\begin{array}{l} P(a,b), \\ P(b,c), \\ P(x,y) \leftarrow P(x,z) \wedge P(z,y). \end{array} \tag{3}$$

The rules of program (1) express that the set P contains a and b, possibly along with other elements. The corresponding completion formula

$$\forall x(P(x) \leftrightarrow x = a \vee x = b) \tag{4}$$

is stronger: it expresses that P is the set $\{a,b\}$. This set is the smallest among the sets containing a and b; in other words, it is the intersection of all sets containing a and b. The condition

x belongs to the intersection of all sets containing a and b

can be expressed by the second-order formula

$$\forall p(p(a) \land p(b) \to p(x)). \tag{5}$$

Consequently, (4) is equivalent to the formula

$$\forall x(P(x) \leftrightarrow \forall p(p(a) \land p(b) \to p(x))). \tag{6}$$

This example motivates the following definition. Let Π be a positive program containing no predicate constants other than P. By $\Pi^\diamond(p)$ we denote the formula obtained by forming the conjunction of the universal closures of the rules of Π and then replacing each occurrence of P in this conjunction by the predicate variable p. The sentence

$$\forall \mathbf{x}(P(\mathbf{x}) \leftrightarrow \forall p(\Pi^\diamond(p) \to p(\mathbf{x}))),$$

where $\mathbf{x}$ is a tuple of distinct object variables, will be called the *stability formula* for P relative to Π. This formula expresses that P is the smallest of the sets p satisfying $\Pi^\diamond(p)$.

Examples. If Π is (1) then $\Pi^\diamond(p)$ is $p(a) \land p(b)$, so that the stability formula in this case is (6). In case of program (2), the stability formula is

$$\forall uv(P(u,v) \leftrightarrow \forall p(p(a,b) \land \forall xy(p(y,x) \to p(x,y)) \to p(u,v))). \tag{7}$$

This formula expresses that P is the smallest of the symmetric binary relations p satisfying $p(a,b)$. Consequently (7) is equivalent to

$$\forall uv(P(u,v) \leftrightarrow (u = a \land v = b) \lor (u = b \land v = a)).$$

In case of program (3), the stability formula is

$$\begin{aligned}&\forall uv(P(u,v) \leftrightarrow \\ &\qquad \forall p(p(a,b) \land p(b,c) \land \forall xyz(p(x,z) \land p(z,y) \to p(x,y)) \to p(u,v))).\end{aligned} \tag{8}$$

This formula expresses that P is the smallest of the transitive binary relations p satisfying $p(a,b)$ and $p(b,c)$, or, in other words, that P is the transitive closure of the relation

$$\{\langle a,b\rangle, \langle b,c\rangle\}.$$

Consequently (8) is equivalent to

$$\forall uv(P(u,v) \leftrightarrow (u = a \land v = b) \lor (u = b \land v = c) \lor (u = a \land v = c)).$$

For any positive program containing no predicate constants other than P, the stability formula for P entails the completion formula for P. For some programs, for instance (1), the two formulas are equivalent to each other. But there are also cases when the stability formula is stronger.

Problem 21. Determine whether stability is equivalent to completion in the case of program (2).

Problem 22. Determine whether stability is equivalent to completion in the case of program (3).

Recall that an *Herbrand interpretation* (of a signature without function constants of nonzero arity) is an interpretation I such that

- the universe of I is the set of all object constants, and
- every object constant is interpreted by I as itself.

An Herbrand interpretation can be identified with the set of ground atoms that are true in it.

For any positive program containing no predicate constants other than P, its *stable model* is the Herbrand interpretation satisfying the stability formula for P. Every positive program has a unique stable model (even a program containing several predicate constants, as will be discussed later.)

Problem 23^e. Consider the program

$$\begin{array}{l} P(a,b), \\ P(b,c), \\ P(d,d), \\ P(x,y) \leftarrow P(y,x), \\ P(x,y) \leftarrow P(x,z) \wedge P(z,y). \end{array}$$

(a) Make the list of ground atoms that are true in the stable model of this program.
(b) Use CLINGO to check that your answer is correct.