Stable Models of Positive Programs, Part 2

Let Π be a positive program, let $P_1, \dots, P_n$ be the list of all predicate constants in its signature, and let $p_1, \dots, p_n$ be distinct predicate variables. By $\Pi^\diamond(p_1, \dots, p_n)$ we denote the formula obtained by forming the conjunction of the universal closures of the rules of Π and then replacing each occurrence of each P_i in this conjunction by the corresponding variable p_i. The *stability formula* for P_i relative to Π is the sentence

$$\forall \mathbf{x}(P_i(\mathbf{x}) \leftrightarrow \forall p_1 \cdots p_n(\Pi^\diamond(p_1, \dots, p_n) \rightarrow p_i(\mathbf{x}))),$$

where $\mathbf{x}$ is a tuple of distinct object variables.

Take, for instance, the program

$$\begin{array}{l} P(a,b), \\ Q(x) \leftarrow P(x,y). \end{array} \tag{1}$$

Its stability formulas are

$$\forall zu(P(z,u) \leftrightarrow \forall pq(p(a,b) \wedge \forall xy(p(x,y) \rightarrow q(x)) \rightarrow p(z,u))) \tag{2}$$

and

$$\forall z(Q(z) \leftrightarrow \forall pq(p(a,b) \wedge \forall xy(p(x,y) \rightarrow q(x)) \rightarrow q(z))). \tag{3}$$

The right-hand side of equivalence (2) can be rewritten as

$$\forall p(p(a,b) \wedge \exists q \forall xy(p(x,y) \rightarrow q(x)) \rightarrow p(z,u)).$$

Since the second conjunctive term in the antecedent is logically valid, this formula is equivalent to

$$\forall p(p(a,b) \rightarrow p(z,u))$$

and consequently to $z = a \wedge u = b$. It follows that stability formula (2) can be equivalently rewritten as

$$\forall zu(P(z,u) \leftrightarrow z = a \wedge u = b).$$

The right-hand side of equivalence (3) can be rewritten as

$$\forall q(\exists p(p(a,b) \wedge \forall xy(p(x,y) \rightarrow q(x))) \rightarrow q(z)).$$

The antecedent of the implication is equivalent to $q(a)$, so that this formula is equivalent to

$$\forall q(q(a) \rightarrow q(z))$$

and consequently to $z = a$. It follows that stability formula (3) can be equivalently rewritten as

$$\forall z(Q(z) \leftrightarrow z = a).$$

The *stable model* of a positive program Π is the Herbrand interpretation satisfying the stability formulas for all predicate constants relative to Π. For instance, the stable model of (1) is

$$\{P(a, b), Q(a)\}.$$

Problem 24. Simplify the stability formulas of the program

$$\begin{array}{l} P(a), \\ Q(b), \\ R(x) \leftarrow P(x), \\ R(x) \leftarrow Q(x) \end{array}$$

and find its stable model.